

Comparison of Multivariate Control Charts
- A Hybrid Approach
for Process Monitoring and Diagnosis

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Outline

- Multivariate Process Control
- Statistical Hypothesis Test and Multivariate Control Charts
- A Hybrid Approach for Process Monitoring and Diagnosis
- Performance Comparison
- Concluding Remarks

Multivariate Process Control

- Hotelling's T^2 chart is one of the most popular control charts for monitoring the mean of a multivariate normal distribution

$$T^2 = (\mathbf{x} - \boldsymbol{\mu}_0)' \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}_0) > \chi_{p,\alpha}^2$$

- Signal Diagnosis - T^2 Decomposition
 - Principle Component Analysis
 - Jackson, 1985
 - Bottom-up Decomposition
 - Mason, Tracy, and Young, 1992, 1995

Multivariate Process Control

- Two alternative approaches
 - Regression-Adjusted Control Chart (Hawkins, 1991, 1993)

where $\|z\|_{\infty} > H$

$$z = \text{diag}(\sigma^{11}, \dots, \sigma^{pp})^{-\frac{1}{2}} \Sigma^{-1} (x - \mu_0)$$

- Simultaneous Confidence Interval (Hayter and Tsui, 1994)

$$\|w\|_{\infty} > C$$

where

$$w = \text{diag}(\sigma_{11}, \dots, \sigma_{pp})^{-\frac{1}{2}} (x - \mu_0)$$

Statistical Hypothesis Test vs. Multivariate Process Control

Testing Multivariate Mean Vector

$$H_0 : \mu = \mu_0 \quad \text{against} \quad \mu \neq \mu_0$$

- When shift direction is known, i.e., $\mu_1 = \mu_0 + \delta d$, the likelihood ratio (LR) statistic

$$U(\mathbf{x}) = \frac{L(\mathbf{x}, \mu_1)}{L(\mathbf{x}, \mu_0)} \sim d' \Sigma^{-1} (\mathbf{x} - \mu_0)$$

provides a uniformly most powerful unbiased test.

Its average run length (ARL) is

$$ARL_{min} = \frac{1}{1 - \Phi(K - \delta) + \Phi(-K - \delta)}$$

Global Tests - Two Principles

- Global Likelihood Ratio Test (GLRT) principle

$H_0 : \mu = \mu_0$ against $H_1 : \mu \in \bigcup S_{\mathbf{d}}^1$,
where $S_{\mathbf{d}}^1 = \{\mu : \mu = \mu_0 + \delta \mathbf{d}\}$.

$$\Lambda(\mathbf{x}) = \frac{\sup_{\mu} L(\mathbf{x}, \mu)}{L(\mathbf{x}, \mu_0)}$$

- Union-Intersection Test (UIT) principle

$H_0 : \mu \in \bigcap S_{\mathbf{d}}^0$ against $H_1 : \mu = \mu_1$,
where $S_{\mathbf{d}}^0 = \{\mu : \mathbf{d}'(\mu - \mu_0) = 0\}$.

For each hypotheses, a rejection region is

$$R_{\mathbf{d}} = \{\mathbf{x} : |\mathbf{d}'(\mathbf{x} - \mu_0)| > c_{\mathbf{d}} \sqrt{\mathbf{d}'\Sigma^{-1}\mathbf{d}}\}.$$

The union of all rejection regions is

$$R = \bigcup R_{\mathbf{d}}$$

Global Tests

Both principles lead to the same monitoring statistic

$$\lambda(\boldsymbol{x}) = (\boldsymbol{x} - \boldsymbol{\mu}_0)' \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu}_0)$$

- Statistical sense: mean vector changes or not
- Variable diagnosis
- Likelihood of which variable(s) experiences changes

Variable Screening

- GLRT Principle

$$H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0 \quad \text{against} \quad H_1 : \boldsymbol{\mu} \in \bigcup_{\wp \in \mathcal{P}} S_{\wp}^1,$$

where $S_{\wp}^1 = \{\boldsymbol{\mu} : \boldsymbol{\mu}_{\wp} \neq \mathbf{0}_{\wp} \text{ and } \boldsymbol{\mu}_{(\wp)} = \mathbf{0}_{(\wp)}\}$.

For each individual hypothesis,

$$\tilde{\lambda}_{\wp}(\mathbf{x}) = \lambda(\mathbf{x}) - \lambda_{(\wp)}(\mathbf{x}) \sim \chi_{|\wp|}^2$$

U^2 statistic - Runger, 1996

This principle reduces to the global T^2 statistic.

Variable Screening

- UIT Principle

$$H_0 : \boldsymbol{\mu} = \bigcap_{\wp \in \mathcal{P}} S_{\wp}^0 \quad \text{against} \quad H_1 : \boldsymbol{\mu} \neq \mathbf{0},$$

$$\text{where } S_{\wp}^0 = \{\boldsymbol{\mu} \mid \boldsymbol{\mu}_{\wp} = \mathbf{0}_{\wp}\},$$

For each individual hypotheses,

$$\lambda_{\wp} = \mathbf{x}_{\wp} \boldsymbol{\Sigma}_{\wp\wp}^{-1} \mathbf{x}_{\wp}$$

The UIT rejection region is

$$R_{\wp} = \{\mathbf{x} : \bar{\lambda} = \lambda_{\wp}(\mathbf{x}) / \chi_{|\wp|, \alpha}^2 > 1\}$$

However,

$$P_0\{\bigcup R_{\wp}\} > \alpha$$

A Hybrid Approach

- Some Observations:
 - No principle dominates the other in detecting all directions of shifts
 - GLRT reduces the variation of a set of variables by using its correlation information with other variables while maintains the signal magnitude
 - GLRT allocates unequal Type I errors to all hypothesis components

Small type I error \rightarrow small group

- Proposal: Assign type I errors to all testing components as equally as possible

Assuming a priori likelihood/probability q_{\wp}^0 of potential changes for each variable subset \wp ,

$$\lambda_{\wp}^{HB}(\mathbf{x}) = \frac{q_{\wp}^0 \tilde{\lambda}_{\wp}(\mathbf{x})}{\chi_{|\wp|, \alpha}^2} > c.$$

A Simplified Procedure

- Using a threshold to screen out un-change variables

$$J^- = \{j : |x_j/\sigma_{jj}| \leq C\}$$

- Segmenting P into two subsets of variables

$$P = J \cup J^-$$

- If $J = P \setminus J^- = \emptyset$, all variables are in-control.
- If $J \neq \emptyset$, monitoring the standardized T^2 statistic of the regression-adjusted variables of J^- ,

$$\lambda^{ADP}(\mathbf{x}) = \frac{\tilde{\lambda}_{\mathbf{J}}(\mathbf{x})}{\chi^2_{|\mathbf{J}|, \alpha}} > H.$$

Properties of the Simplified Approach

- When $C = 0$, it is exactly the T^2 chart.
- When $C \geq C_{HT}$, it is Hayter and Tsui's M chart.
- When $0 < C < C_{HT}$, it is something similar to the hybrid chart.

Performance Comparison

- Benchmark: UMP test given shift direction

Relative Efficiency:

$$RE = \frac{ARL}{ARL_{min}}$$

- Mean Shift Magnitude:
 - Quality Measure

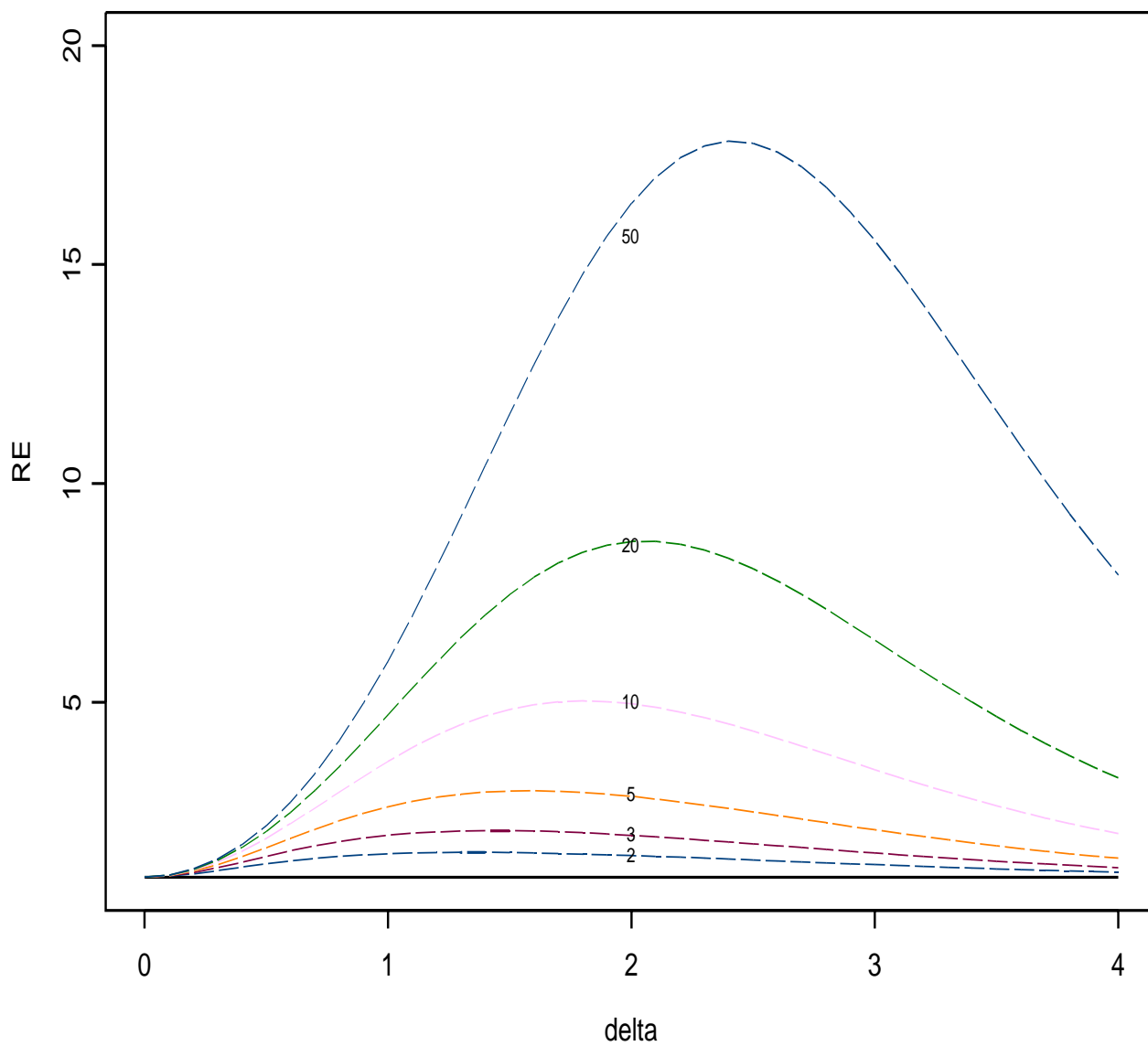
$$(x - \mu_0)' (x - \mu_0)$$

- Statistical Measure

$$(x - \mu_0)' \Sigma^{-1} (x - \mu_0)$$

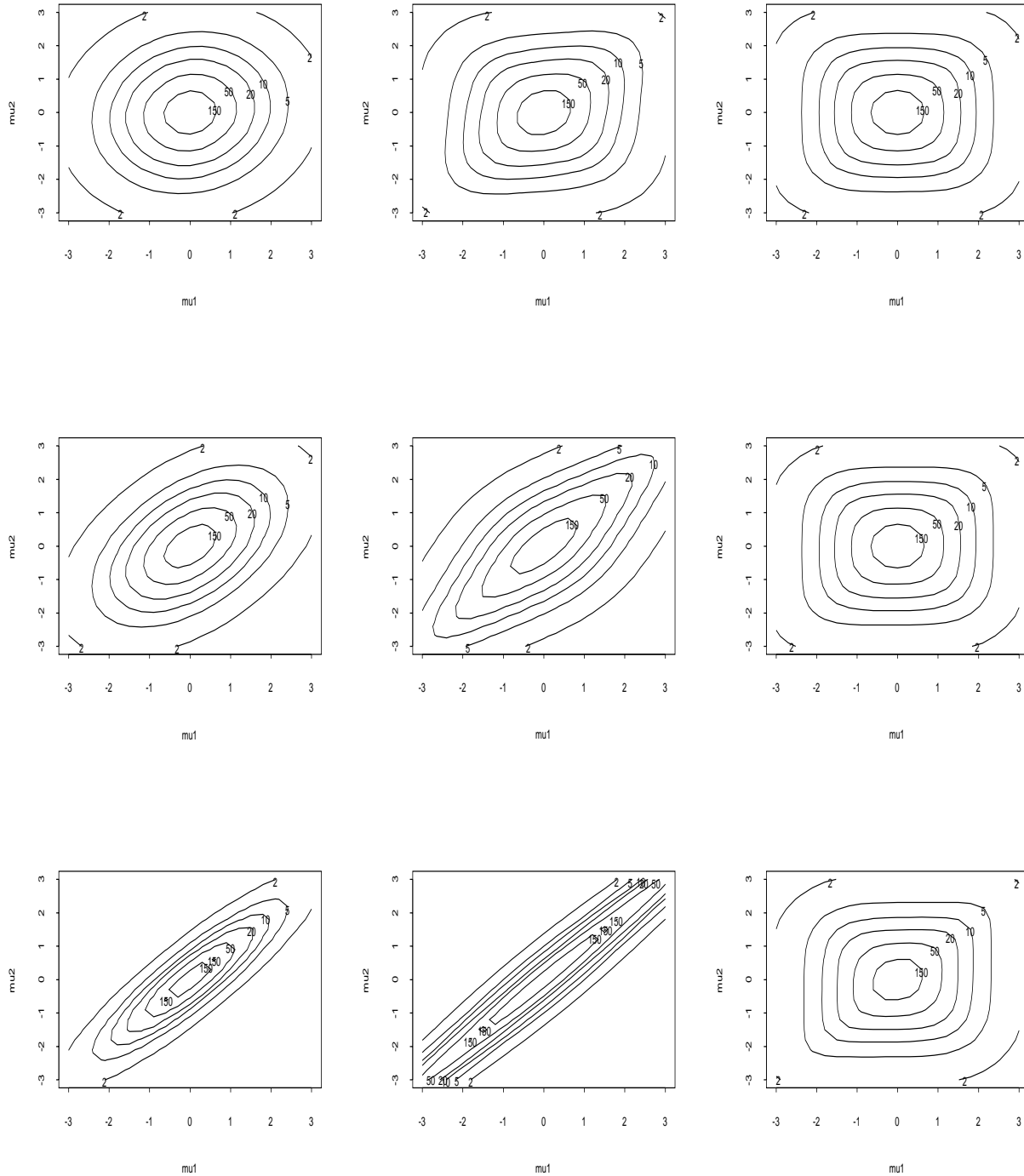
RE - Curse of Dimensionality

T^2 chart



ARL - Quality View

T^2 , Regression-Adjusted, M charts



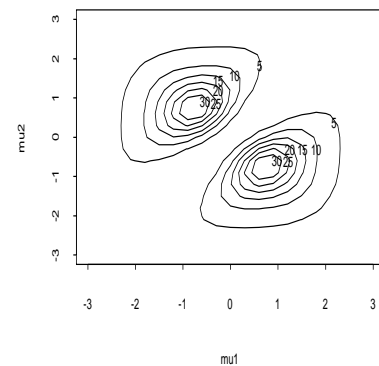
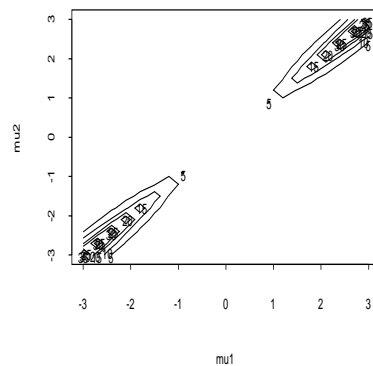
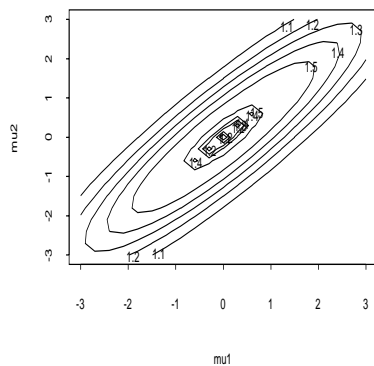
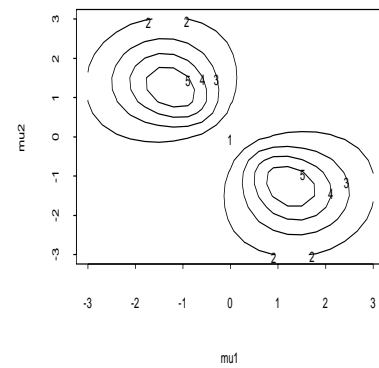
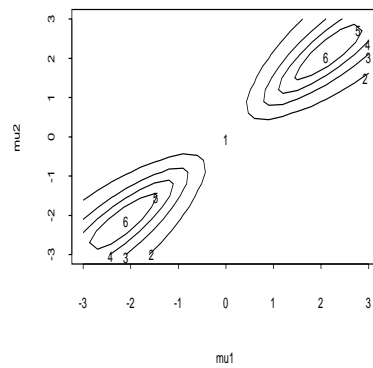
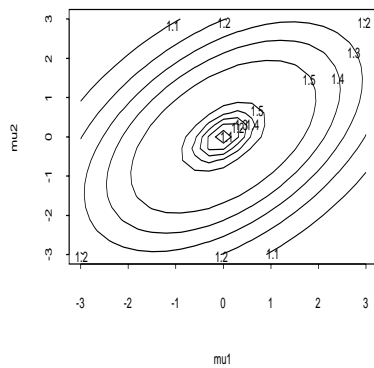
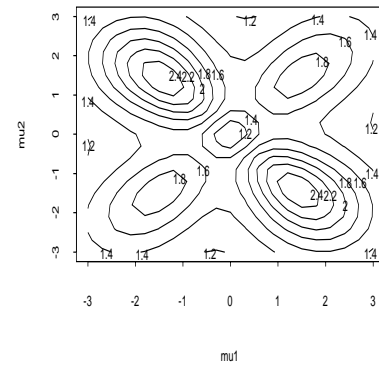
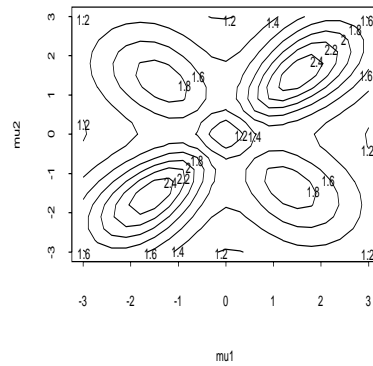
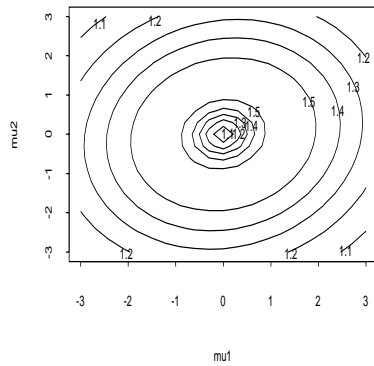
(a) T^2 chart

(b) Regression-Adjusted chart

(c) M chart

RE - Contour Plots of Statistical View

T^2 , Regression-Adjusted, M charts



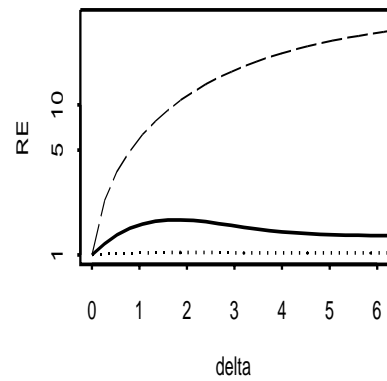
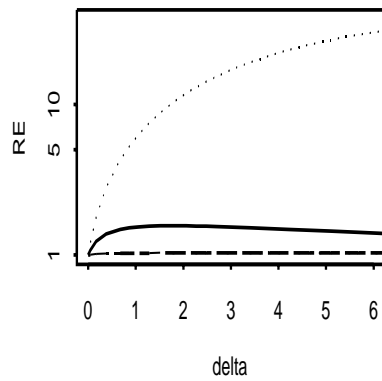
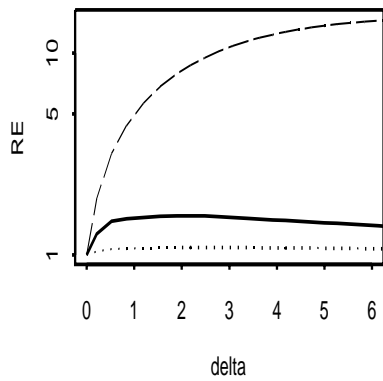
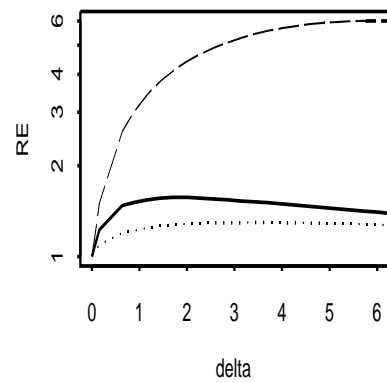
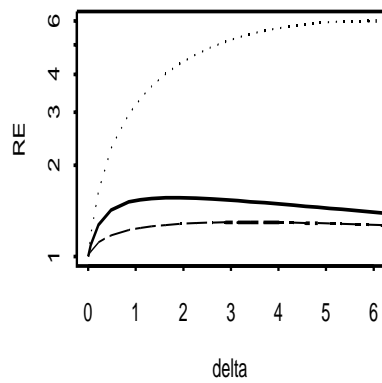
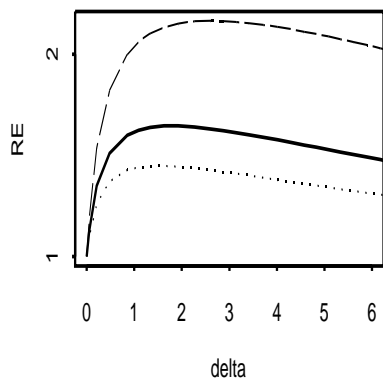
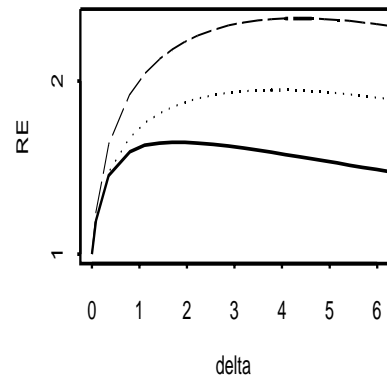
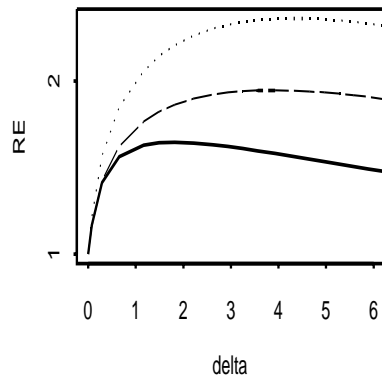
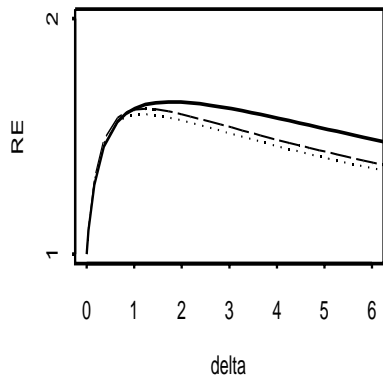
(a) T^2 chart

(b) Regression-Adjusted chart

(c) M chart

RE - Some Special Cases

T^2 , Regression-Adjusted, M charts



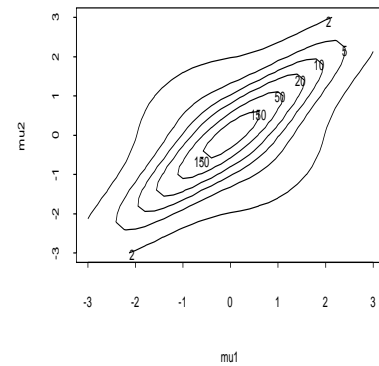
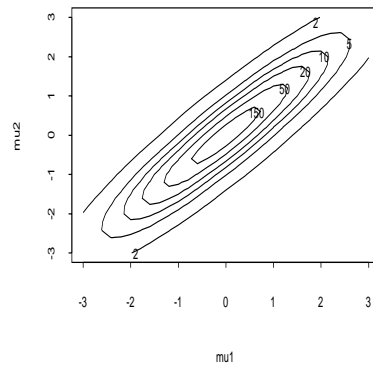
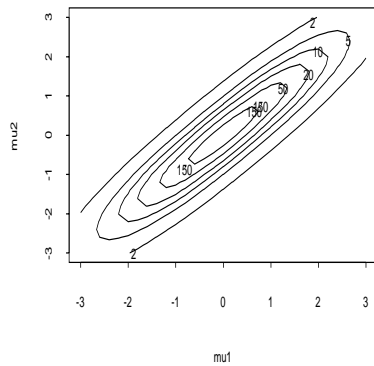
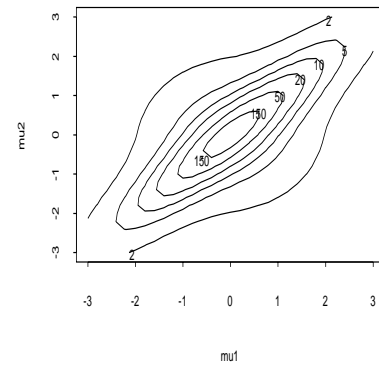
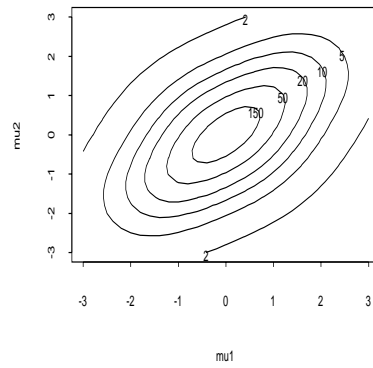
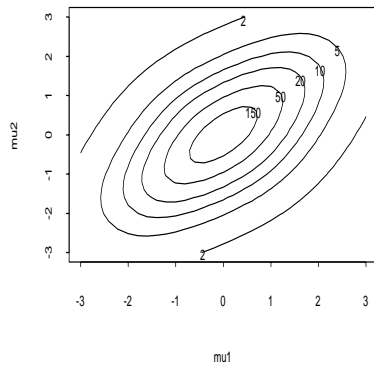
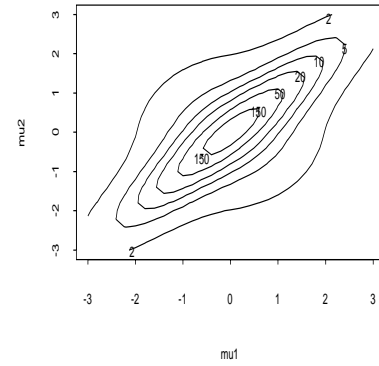
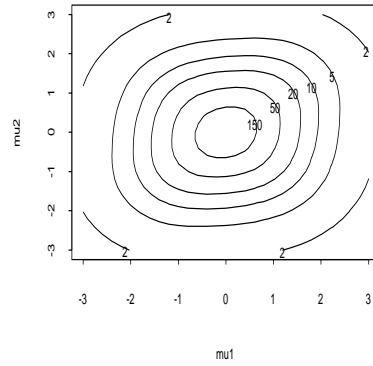
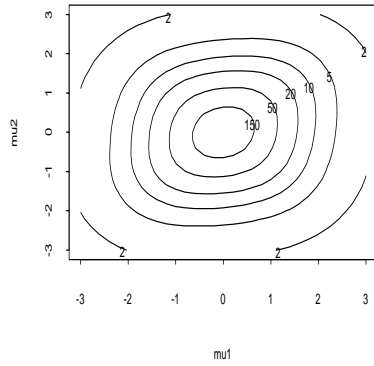
(a) $\mu_1 = 0$

(b) $\mu_1 = \mu_2$

(c) $\mu_1 = -\mu_2$

ARL - Quality View

Hybrid charts

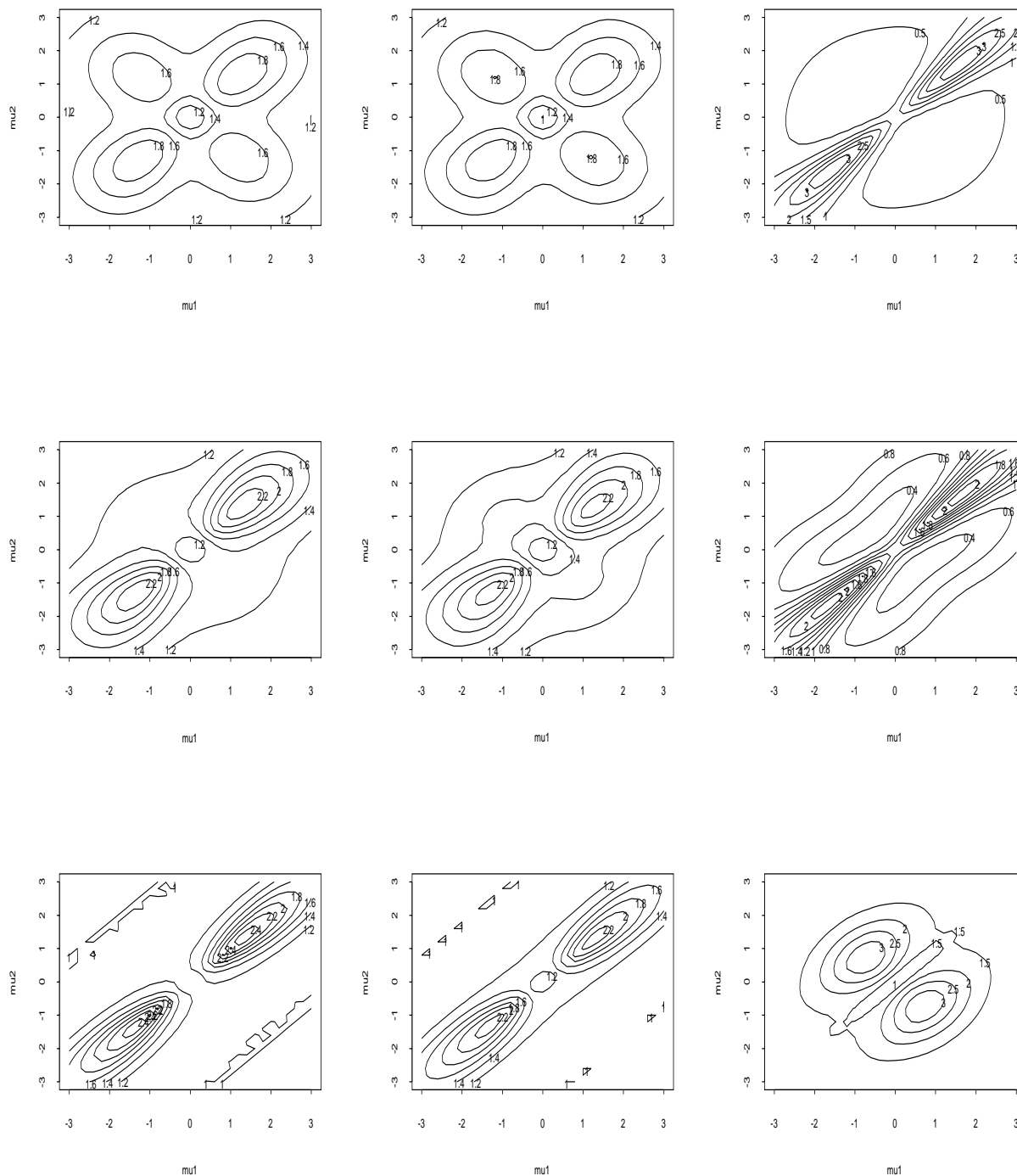


(a) Hybrid chart

(b) Adaptive chart $C = 1$

(c) Adaptive chart $C = 2$

RE - Contour Plots of Statistical View Hybrid charts



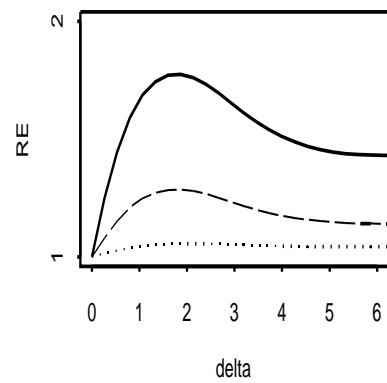
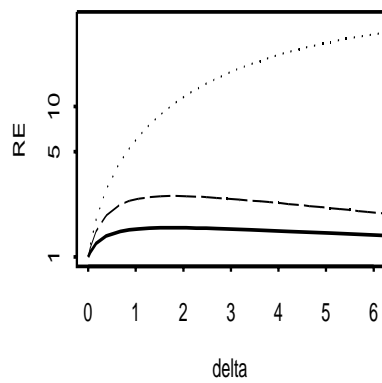
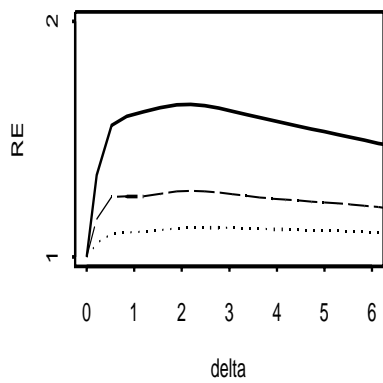
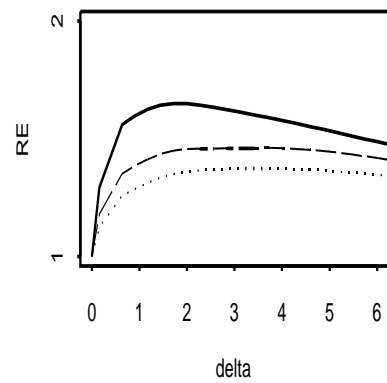
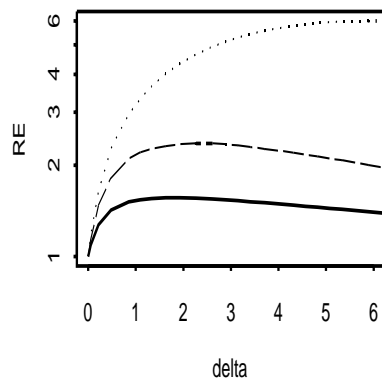
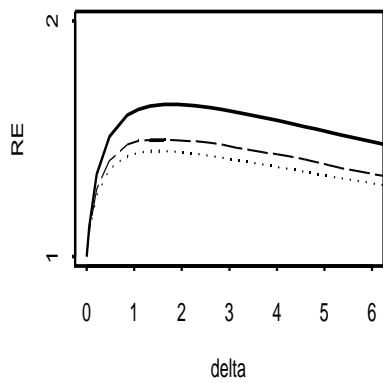
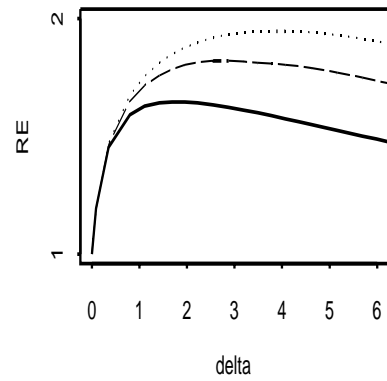
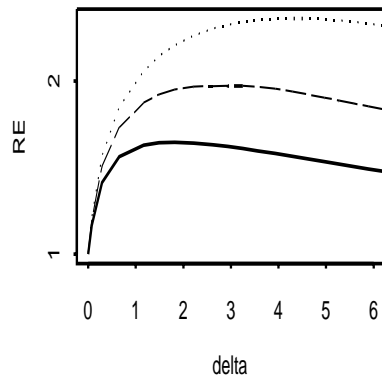
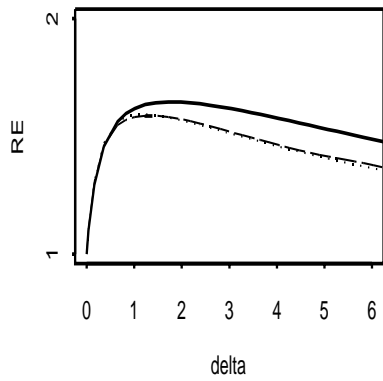
(a) Hybrid chart

(b) Adaptive chart $C = 1$

(c) Adaptive chart $C = 2$

RE - Some Special Cases

T^2 , Regression-Adjusted, Hybrid charts



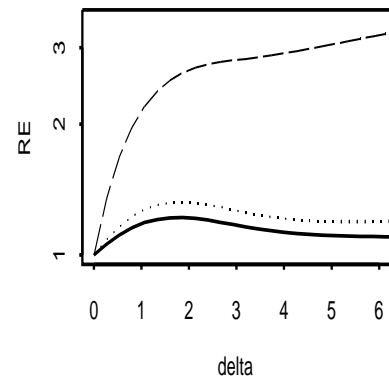
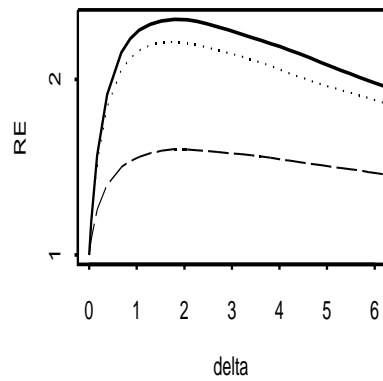
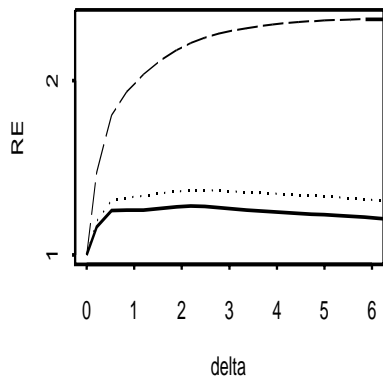
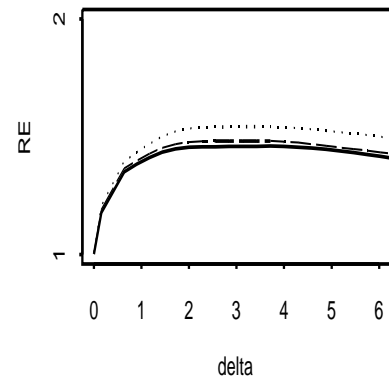
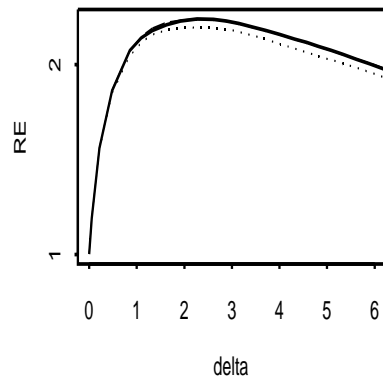
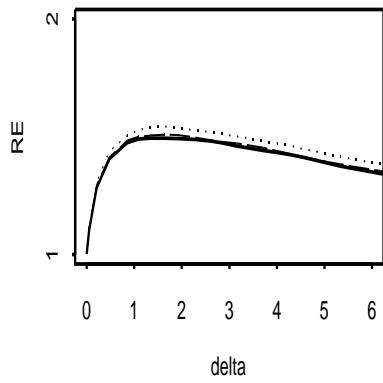
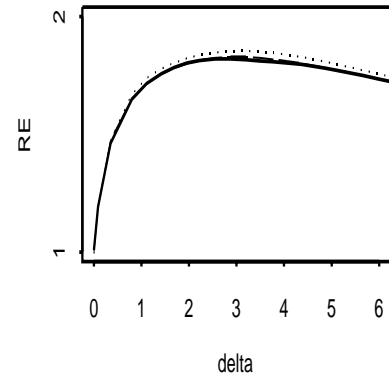
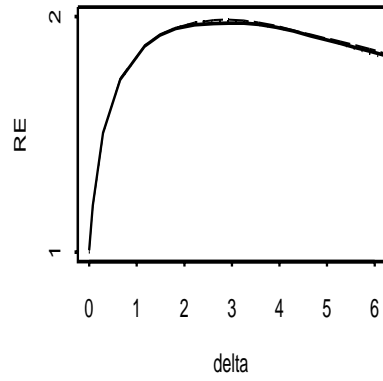
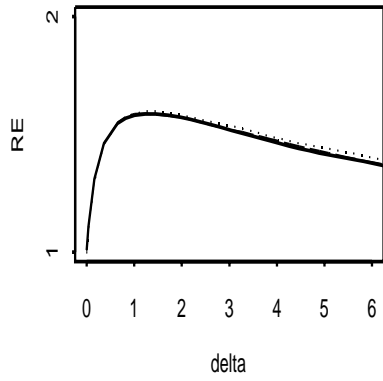
(a) $\mu_1 = 0$

(b) $\mu_1 = \mu_2$

(c) $\mu_1 = -\mu_2$

RE - Some Special Cases

Hybrid charts and Simplification



(a) $\mu_1 = 0$

(b) $\mu_1 = \mu_2$

(c) $\mu_1 = -\mu_2$

Conclusions and Future Research

- A hybrid method is proposed based on the GLRT and UIT principles to integrate multivariate process monitoring and diagnosis.
- An adaptive approach is introduced to simplify the hybrid method in practice.
- The diagnosis of signals from the adaptive approach is simpler than the T^2 decomposition. Needs further investigations.