Optimal Compound Orthogonal Arrays and Single Arrays for Robust Parameter Design Experiments

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Outline

- Introduction
- Optimal Strong Compound Orthogonal Array
- Optimal Economical Single Array
- Compound Array versus Single Array
- Conclusion

Robust Parameter Design (RPD)

- Quality improvement via variation reduction (Taguchi, 1986)
- Two types of factors: control and noise factors
- Key idea:
  Explore the effects of control factors, noise factors, and their interactions, and choose control settings to simultaneously optimize system mean response and reduce variation (Wu and Hamada, 2000)
- Both control and noise factors are systemetically varied during experimentation.

Experiment and Modeling Strategies

- Cross array or inner-outer product array orginally proposed by Taguchi:
  a cross product of a design (array) for control factors and a design (array) for noise factors

  At each fixed control setting, response mean and dispersion can be calculated.

  Use location-dispersion modeling to identify best settings of control factors

- Single array or combined array proposed by Lucas, Welch et al.:
  one single design (array) to accomodate both types of factors.

  Use response modeling to model response as a function of control effects, noise effects and their interactions

  Derive mean response model and variance transmitted model for optimization.

Optimal Experimental Plans

- Optimal experimental plans for robust parameter design: still an unsettled open problem.
- Compound orthogonal array as a generalization of cross array (Rosenbaum, 1994, 1996 and 1998)

  not requiring rigid crossing structure
- Optimal single array
  - Opitmality criterion based on modified wordlength patterns (Bingham and Sitter, 2003)
  - General framework based on new effects ordering principle and weighted combination of wordlength patterns (Zhu and Wu, 2003)

  not sensitive or too complex
- Simple and direct approaches are in order to construct optimal experimental plans for robust parameter designs.

2^{l-p} Designs

- Regular two-level fractional factorial designs generated by m independent defining relations (words)
- Defining contrast subgroup $\mathcal{G}$.
- Wordlength pattern:

$$W = (W_1, W_2, \ldots, W_l)$$

  W_i: the number of defining words of length i in $\mathcal{G}$.
- Resolution is the smallest i such that $W_i > 0$, and

$$\text{strength} = \text{resolution} - 1$$

- Minimum aberration (MA) design sequentially minimizes W_i and considered to be optimal.
- An effect is clear if it is not aliased with main and two factor interactions.

$2^{(l_1+l_2)-p}$ Designs with Two Groups of Factors

- l_1 factors belong to Group I and l_2 factors belong to Group II.
- p independent defining words and defining contrast subgroup $\mathcal{G}$.
- Wordtype Pattern matrix $(A_{i.j})_{0\leq i\leq l_1;0\leq j\leq l_2}$ where

  $A_{i.j}$: number of words in $\mathcal{G}$ involving i Group I factos and j Group II factors

- $A_{i.j}$ can be arranged into a sequence

$$W_t = (A_{3.0}, A_{2.1}, A_{1.2}, A_{0.3}, A_{4.0}, \ldots)$$

- For robust parameter design, control factors form Group I (l_c) and noise factors form Group II (l_n).
- Both cross array and single array can be viewed as $2^{(l_c+l_n)-p}$ designs

Roadmap for constructing optimal plans for RPD

- For fixed experiment size $N = 2^k$, define

$$\mathcal{S}(2^k) = \{(i, j) : \lceil \log_2(i+1) \rceil + \lceil \log_2(j+1) \rceil \le k\}.$$

- for the cases where cross arrays exist, that is, $(l_c, l_n) \in \mathcal{S}(2^k)$, propose and use optimality criterion to select optimal compound orthogoal array
- for the cases where cross arrays do not exist, that is, $(l_c, l_n) \notin \mathcal{S}(2^k)$, propose and use criterion to select optimal economical single array

- For the former cases, select optimal general single arrays according to various criteria and compare them with optimal compound orthogonal arrays to decide which to use.

Strong Compound Orthogonal Array (SCOA)

- A $2^{(l_c+l_n)-p}$ is a strong compound orthogonal array if $A_{i.1} = A_{i.2} = 0$ for $1 \le i \le l_c$.

  Example: four control factos (A, B, C, D) and three noise factors (a, b, c):

  $$d : I = ABCD = ABabc = CDabc$$

  At each fixed setting of A, B, C and D, the settings of a, b and c form a design defined by either by $abc = 1$ or $abc = -1$ with strength 2.

- Noise arrays in a strong compound orthogonal array have strength at least 2.

Order of Aliaising Severity and W_c-Aberration

- A_{i_1,j_1} is more severe than A_{i_2,j_2}, denoted by $A_{i_1,j_1} \lhd A_{i_2,j_2}$, if

  (i) $i_1 + j_1 < i_2 + j_2$; or

  (ii) $\mid i_1 - j_1 \mid < \mid i_2 - j_2 \mid$ and $i_1 + j_1 = i_2 + j_2$; or

  (iii) $i_1 > i_2$ and $|i_1 - j_1| = |i_2 - j_2|$ and $i_1 + j_1 = i_2 + j_2$.

- W_c-sequence

$$W_c = (A_{3.0}, A_{0.3}, A_{1.3}, A_{4.0}, A_{0.4}, A_{2.3}, A_{1.4}, \ldots)$$

- d_1 is said to have less W_c-aberration than d_2 if $W_c(d_1)$ and $W_c(d_2)$ first differ at A_{i_0,j_0} and $A_{i_0,j_0}(d_1) < A_{i_0,j_0}(d_2)$.

- Strong compound orthogonal arrays with minimum W_c-aberration are optimal

Tables of Optimal SCOAs with $l_n \geq 3$

Design	generator	strength	clear effects
16-run:			
...	...	...	...
$(3,3)^{\circ}$	ABC $Aabc$	(2, 2, 2)	(0, 3, 0, 6, 0)
...	...	...	...
32-run:			
...	...	...	...
$(2,5)^{\star}$	$abcd$ $ABabe$	(2, 2, 3)	(2, 5, 1, 10, 4)
$(3,4)^{\star}$	ABC $Aabcd$	(2, 3, 2)	(0, 4, 0, 12, 6)
$(4,3)^{\star}$	$ABCD$ $ABabc$	(3, 2, 3)	(4, 3, 0, 12, 3)
...	...	...	...
64-run:			
...	...	...	...
$(3,6)^{\star}$	$abce$ $ACabd$ $Bacdf$	(3, 2, 3)	(3, 6, 3, 18, 9)
$(4,5)^{\star}$	$ABCD$ $ABabd$ $ACace$	(3, 2, 3)	(4, 5, 0, 20, 10)
$(5,4)^{\star}$	ABD ACE $BCabcd$	(2, 3, 2)	(0, 4, 0, 20, 6)
...	...	...	...

Economical Single Arrays (ESA)

- A single array $2^{(l_c+l_n)-p}$ with $(l_c, l_n) \notin \mathcal{S}(N)$ is called an Economical Single Arrays:
  - Experiment is not big enough to allow crossing structure
  - Only response modeling can be used to analyze data
  - Must prioritize crucial effects such as control main effects, control-by-noise interactions.
  - Have to discriminate against unimportant effects such as noise effects
- Different trade-off schemes lead to different criteria

W_s Sequence and W_s-Aberration

- Recall that W_c sequence does not contain $A_{i.1}$ and $A_{i.2}$.
- Properly including $A_{i.1}$ and $A_{i.2}$ in W_c leads to

$$W_s = (\mathbf{A_{2.1}}, \mathbf{A_{1.2}}, A_{3.0}, A_{0.3}, A_{2.2}, \mathbf{A_{3.1}}, A_{1.3}, A_{4.0}, A_{0.4}, \mathbf{A_{3.2}}, A_{2.3}, \ldots).$$

- W_s partially perserves the hierarchical ordering principle
- W_s-aberration and minimum W_s-aberration criterion can be proposed.
- Due to limited capacity in ESA, W_s is too restrictive.

Split Wordtype Pattern and (W_{sm}, W_{sn})-Aberration

- Split W_s into two separate sequences

$$W_{sm} = (A_{2.1}, A_{1.2}, A_{3.0}, A_{2.2}, A_{3.1}, A_{1.3}, A_{4.0}, A_{3.2}, A_{2.3}, \ldots),$$

$$W_{sn} = (A_{0.3}, A_{0.4}, A_{0.5}, A_{0.6}. \ldots).$$

- Note that W_{sn} contains patterns involving noise factors only.
- Select ESAs frist according to W_{sm}, then further use W_{sn}, or in short, according to (W_{sm}, W_{sn})-aberration
- Advantage: the selected ESAs possess clear combinatorial structure
- Disadvantage: the trade-off between important and unimportant effects may be too extreme

Shifted Wordtype Pattern and W_{ss}-Aberration

- Start with W_s, shift the patterns involving only noise factors rightward to proper positions

$$W_{ss} = (A_{2.1}, A_{1.2}, \dots, A_{4.0}, A_{3.2}, \mathbf{A_{0.3}}, A_{2.3}, A_{4.1}, \dots, A_{4.2}, \mathbf{A_{0.4}} \dots)$$

- W_{ss}-aberration and minimum W_{ss}-aberration can be defined.
- Other possible milder trade-off schemes could also be used.
- Often lead to the same optimal ESA as the split wordtype patterns.

Tables of Optimal ESAs with minimum (W_{sm}, W_{sn})-Aberration

Design	generator	clear 2fi
16-run:		
$(2,4)$	$abc\ ABad$	(2, 1, 0, 4, 2)
$(2,5)$	$abd\ ace\ ABbc$	(2, 0, 0, 2, 0)
$(2,6)$	$abd\ ace\ bcf\ ABabc$	(2, 0, 0, 0, 0)
32-run:		
$\ldots$	$\ldots$	$\ldots$
$(4,6)$	$abd\ ace\ bcf\ ACabc\ ABDa$	(4, 0, 0, 8, 0)
$(5,5)$	$abd\ ace\ ACbc\ BDbc\ ABEa$	(5, 0, 0, 4, 0)
$(6,4)$	$abc\ ABDa\ ACEa\ BCFa\ ABCbd$	(6, 1, 0, 6, 2)
$(2,9)$	$abe\ acf\ bcg\ abch\ adi\ ABbd$	(2, 0, 0, 10, 0)
$\ldots$	$\ldots$	$\ldots$
64-run:		
$(4,8)$	$abe\ acf\ bcg\ Aabch\ ACad\ BDabcd$	(4, 2, 5, 20, 4)
$(8,4)$	$Cabc\ Dabd\ AEacd\ BFacd\ ABGab\ ABHbcd$	(8, 4, 12, 24, 0)
$(4,9)$	$abe\ acf\ bcg\ adh\ Aabci\ ACbd\ ABDacd$	(4, 1, 5, 20, 2)
$(5,8)$	$abd\ ace\ bcf\ abcg\ ABDa\ ACEb\ BCch$	(5, 1, 2, 26, 5)
$\ldots$	$\ldots$	$\ldots$

General Single Arrays

- When SCOA arrays exists, general single arrays can also be considered.
- It is not conclusive which criterion we should use for selecting optimal genenal single array
- Use all W_s-, (W_{sm}, W_{sn})- and W_{ss}-aberrations and compare the optimal designs.
- Other considerations may be used to determine which optimal design to use in practice.

Table of General Optimal Single Arrays

Design	Defining Generators	W_s	(W_{sm}, W_{sn})	W_{ss}	COA	MA
16-Run:						
...	...	...	...	...	...	...
(3, 3)	$ABCa\ ABbc$	$\surd$				$\surd$
(3, 3)	$ABCa\ abc$		$\surd$	$\surd$		
...	...	...	...	...	...	...
32-Run:						
...	...	...	...	...	...	...
$(3, 4)$	$abcd\ ABCab$	$\surd$				$\surd$
$(4, 3)$	$ABCD\ ABabc$	$\surd$			$\surd$	$\surd$
$(4, 3)$	$abc\ ABCDa$		$\surd$	$\surd$		
$(5, 2)$	$ABCD\ ABEab$	$\surd$	$\surd$	$\surd$		$\surd$
...	...	...	...	...	...	...
64-run:						
$(3, 5)$	$ABabd\ Cabce$	$\surd$			$\surd$	$\surd$
$(4, 4)$	$Aabcd\ ABCDa$	$\surd$				$\surd$
...	...	...	...	...	...	...

Conclusion: An Example

- Four control factors (A, B, C, D) and three noise factors (a, b, c) in a RPD experiment
- If can afford to conduct 64 runs, the optimal SCOA is genereated by

$$d_1 : I = ABCD = ABabd = ACace;$$

[The optimal general single array is generated by

$$d_2 : I = abce = ACabd = ABCDbc$$

if crossing structure is not crucial, d_2 is more attractive.

- If can afford to conduct 32 runs only, then

32-run SCOA does not exists

optimal economical single array is generated by

$$d_3 : I = ABac = ABbd = Aabd = BCDab$$

- The designs are obtained from the complete tables.