Applications of Using Quaternary Codes to Nonregular Designs

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Notations and Definitions

Let D be a design of N runs and n factors.

J-characteristics (J_k) of D:

Let $s = \{x_1, \cdots\cdots, x_k\}$ a subset of k columns of D. Then

$$J_k(s) = \left| \sum_{i=1}^{N} \prod_{j=1}^{k} x_{ij} \right| \qquad 0 \le J_k(s) \le N$$

When D is a regular design, $J_k(s) \in \{0, N\}$

Example:

A	B	C
-1	-1	-1
-1	$+1$	$+1$
$+1$	-1	$+1$
$+1$	$+1$	-1

$A: J_1(A) = (-1)+(-1)+(+1)+(+1) = 0$

$B: J_1(B) = (-1)+(+1)+(-1)+(+1) = 0$

$C: J_1(C) = (-1)+(+1)+(+1)+(-1) = 0$

$\rightarrow J_1(s) = J_1(\{A, B, C\}) = \{0, 0, 0\}$

$AB: J_2(AB) = (-1)(-1)+(-1)(+1)+(+1)(-1)+(+1)(+1) = 0$

$AC: J_2(AC) = (-1)(-1)+(-1)(+1)+(+1)(+1)+(+1)(-1) = 0$

$BC: J_2(BC) = (-1)(-1)+(+1)(+1)+(-1)(+1)+(+1)(-1) = 0$

$\rightarrow J_2(s) = J_2(\{AB, AC, BC\}) = \{0, 0, 0\}$

$ABC: J_3(ABC) = (-1)(-1)(-1)+(-1)(+1)(+1)+(+1)(-1)(+1)+(+1)(+1)(-1) = -4$

$\rightarrow J_3(s) = J_3(\{ABC\}) = \{4\}$

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Notations and Definitions

Generalized Resolution of D:

Suppose that r is the smallest integer such that $\max_{|s|=r} J_r(s) > 0$. Then we define

$$R(D) = r + \left(1 - \max_{|s|=k} \frac{J_k(s)}{N}\right)$$

the generalized resolution of D.

Example:

A	B	C
−1	−1	−1
−1	+1	+1
+1	−1	+1
+1	+1	−1

$J_1(s) = \{0,0,0\}$ $\quad r = 3$

$J_2(s) = \{0,0,0\}$ $\quad N = 4$

$J_3(s) = \{4\}$ $\quad \max J_k(s) = 4$

$$R(D) = r + \left(1 - \max_{|s|=k} \frac{J_k(s)}{N}\right) = 3 + \frac{4}{4} = 3.0$$

Basics on Quaternary (Z_4) Codes

Algorithms of using Z_4-codes to generate nonregular designs:

1. Let G (generator matrix) be an r x n full-rank matrix over Z_4.
2. A 4-level design C is formed based on G.
3. Applying **Gray map**, a 2-level design D is formed based on C.

Bijection Transformation of Gray Map:

Z_4	0	1	2	3
	↓	↓	↓	↓
Z_2^2	00	01	11	10

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Basics on Quaternary (Z_4) Codes

Example:

1. Given a 2 x 4 generator matrix:

$$G = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

2. A 4-level design C is formed based on G.

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 3 & 3 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 3 \\ 1 & 2 & 0 & 1 \\ 1 & 3 & 1 & 3 \\ 2 & 0 & 0 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 2 & 2 & 2 \\ 2 & 3 & 3 & 0 \\ 3 & 0 & 2 & 3 \\ 3 & 1 & 3 & 1 \\ 3 & 2 & 0 & 3 \\ 3 & 3 & 1 & 1 \end{pmatrix}$$

3. A 2-level design D is formed based on C through Gray map.

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Z_4	0	1	2	3
	↓	↓	↓	↓
Z_2^2	00	01	10	11

Generalized Resolution = 4.0
(GOOD DESIGN)

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Summaries on Previous Results

Nordstrom and Robinson Code:

Hedayat, Sloane and Stufken (1999); Xu (2005)

It can form a 256 runs, 16 columns designs.

- Generalized Resolution = 6.5
- Minimum Runs to achieve this in Regular Design = 512 runs

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 3 \end{pmatrix}$$

Design Construction with GR = 3.5:

Xu and Wong (2005)

Rule of Eliminating Columns:

1. Delete columns that do not contain any 1's
2. Delete columns whose first non-zero and non-two entry are 3's.

Then the rest of the columns will form a generating matrix with GR=3.5.

Summaries on Previous Results

<u>Example - 16 runs design:</u>

From the list of all possible columns:

Step #1: Delete columns that do not contain any 1’s

Step #2: Delete columns whose first non-zero and non-two entry are 3’s.

Result:

1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
0	0	0	1	1	1	1	2	2	2	2	3	3	3	3

Generalized Resolution = 3.5

New Applications I:

Design Construction with Resolution 4.0

Additional Step on Previous Algorithm:

Result from Previous Steps:

Step #3: Delete columns whose sum of columns entries are even.

Result:

1			0	1	2			1				1		
0			1	1	1			2				3		

$$G = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Generalized Resolution = 4.0

New Applications I:

Design Construction with Resolution 4.0

Maximum Columns of Design with GR=4.0:

# Rows in G	# Columns in G	Dimension of D	Resolution
2	4	16 x 8	4.0
3	16	64 x 32	4.0
4	64	256 x 128	4.0
:	:	:	:
N	4^{n-1}	$4^n \times 4^n/2$	4.0

This table suggests that the maximum columns of design with GR =4.0 is N/2 with N runs.

Previous complete search algorithm breaks down when N>256.

This construction generalizes the case for higher run sizes.

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New Applications II:

Fast Search Results on Optimal 2^{n-2} Designs

General Structure for Optimal (over Z_4) 2^{n-2} Designs:

$$\left[\begin{array}{c|c} \left(\quad I_k \quad \right) & \begin{array}{c} \left.\begin{array}{c} 2 \\ \vdots \\ 2 \end{array}\right\} b \\ \left.\begin{array}{c} 1 \\ \vdots \\ \vdots \\ 1 \end{array}\right\} a \end{array} \end{array}\right]$$

Notation:

G = I_k(a,b) design

where

k = size of Identity matrix

a = number of 1's in the last column

b = number of 2's in the last column

Optimality is measured in terms of the Generalized Resolution.

Example:

$$I_2(1,1) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad GR = 4.0$$

$$I_3(2,1) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad GR = 5.5$$

$$I_4(2,2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad GR = 6.0$$

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New Applications II:

Fast Search Results on Optimal 2^{n-2} Designs

Fast Calculation on Generalized Resolution of Even Design:

1. If b ≥ a, then GR = 2(a+1).
2. If b < a, then GR = min{2b+a+1 , 2(a+1)} + decimals.
3. $decimal = 1 - 2^{-\frac{a}{2} \bmod 1}$

Example:

$$I_5(2,3) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$I_4(3,1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

GR = min{2b+a+1,2(a+1)}=min{6,8}=6.x

GR = 2(a+1) = 2(2+1) = 6.0

decimal = 1–2^(-1) = 0.5

GR = 6.0+0.5=6.5

New Applications II:

Fast Search Results on Optimal 2^{n-2} Designs

Fast Calculation on Generalized Resolution of Odd Design:

1. If b ≥ a, then GR = 2(a+1).
2. If b < a, then GR = min{2b+a , 2(a+1)} + decimals.
3. $decimal = 1 - 2^{-\frac{a}{2} \bmod 1}$

Example:

$$I_5(2,3) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$I_4(3,1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

GR = min{2b+a+1,2(a+1)}=min{5,8}=5.x

GR = 2(a+1) = 2(2+1) = 6.0

decimal = 1–2^(-1) = 0.5

GR = 5.0+0.5=5.5

New Applications II:

Fast Search Results on Optimal 2^{n-2} Designs

Search Results on Optimal Designs of different runs:

k	2	2	3	3	4	4	5	5	6	6	7
#col	6	6	8	8	10	10	12	12	14	14	16
Run	8	16	32	64	128	256	512	1024	2048	4096	8192
(a,b)	(1,1)	(1,1)	(2,1)	(2,1)	(2,2)	(3,1)	(3,2)	(3,2)	(4,2)	(4,2)	(4,3)
GR	3.0	4.0	4.5	5.5	6.0	6.5	7.75	8.0	8.75	9.75	10.0

This search algorithm bypasses the actual calculation of J-characteristics wordlength pattern

→ Computer time on searching the optimal designs of particular run size is much less than the traditional methods

→ Optimality of exceptionally large design is possible

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Future Extensions

1. Construction of Designs with higher Resolutions.

   → Algorithm on GR = 4.5 Construction

   → Designs Constructions with properties similar to Nordstrom-Robinson Design.

2. Extensions of Fast Search Algorithms.

   → Optimality of 2^{n-4} Design.

   → Ultimate optimization of designs.

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Conclusion

1. An Introduction on the basics of the construction of two-level nonregular design using quaternary codes are given.
2. Previous research shows that the designs generated have better minimum aberration than regular designs in terms of resolutions and aberration.
3. A new construction method generates the design with GR=4.0, an extension from the previous GR=3.5.
4. A new search method provides some new results to fast search for the optimal designs with 2^{n-2} runs.
5. Future extensions on the design construction, fast search algorithm are suggested.

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