Adaptive OFAT Applied to Robust Parameter Design

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Outline

- Background and motivation
- A mathematical model and theorems
- Robust Design
  - Model-based evaluation
  - Case studies
- Conclusions

Adaptation in Experimentation

- Most industrial experimentation has a characteristic of "immediacy"
- Each set of experimental runs supply a basis for deciding the next
- "What has been made clear by my industrial experience … there should be more studies of statistics from the dynamic point of view"

Box, G.E.P., 1999, "Statistics as a Catalyst to Learning by Scientific Method Part II – A Discussion," *Journal of Quality Technology* 31(1)16-29.

OFAT: A Starting Point for Investigation

"Some scientists do their experimental work in single steps. They hope to learn something from each run … they see and react to data more rapidly …"

"…Such experiments are economical"

"…May give biased estimates"

"If he has in fact found out a good deal by his methods, it must be true that the effects are at least three or four times his average random error per trial."

Daniel, Cuthbert, 1973, "One-at-a-Time Plans", *Journal of the American Statistical Association*, vol. 68, no. 342, pp. 353-360.

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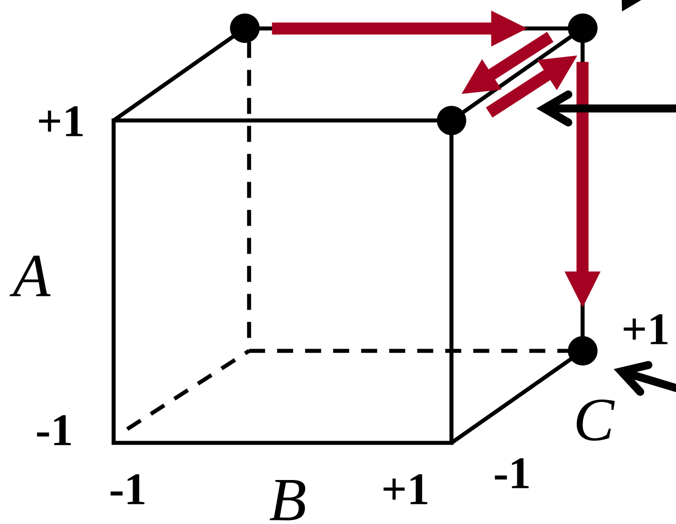
Frey, D. D., and H. Wang, 2006, "Adaptive One-Factor-at-a-Time Experimentation and Expected Value of Improvement", accepted to *Technometrics.*

A Simple Adaptive Version of OFAT

Do an experiment

Change
one factor

If there is an *apparent* improvement, retain the change



If the response gets worse, go back to the previous state

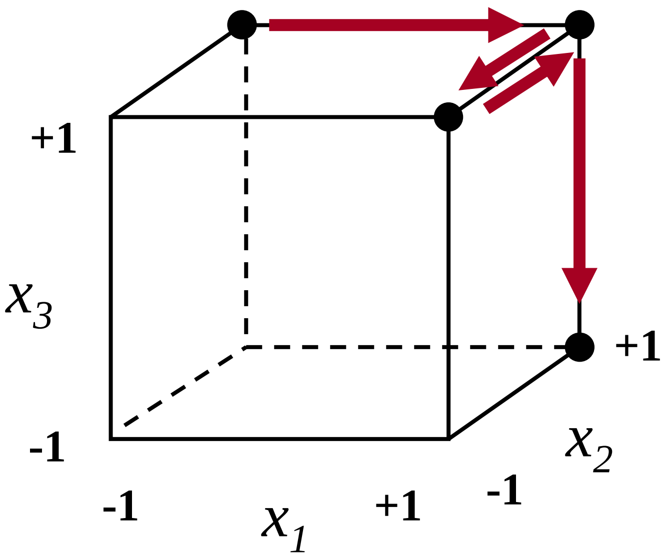
Stop after every factor has been changed exactly once

Mathematical Model – Adaptive Method

$$O_0 = y(\tilde{x}_1, \tilde{x}_2, \ldots \tilde{x}_n)$$

$$O_1 = y(-\tilde{x}_1, \tilde{x}_2, \ldots \tilde{x}_n)$$

$$x_1^* = \tilde{x}_1 sign\{O_0 - O_1\}$$



$$O_i = y(x_1^*, \ldots x_{i-1}^*, -\tilde{x}_i, \tilde{x}_{i+1}, \ldots \tilde{x}_n)$$

$$x_i^* = \tilde{x}_i sign\{\max(O_0, O_1, \ldots O_{i-1}) - O_i\}$$

$$\boxed{E[y(x_1^*, \ldots x_{i-1}^*, \tilde{x}_i, \tilde{x}_{i+1}, \ldots \tilde{x}_n)] = ?}$$

What is the expected improvement at this stage of the process?

Mathematical Model – Factor Effects

- A model of the population of systems is:

$$y(x_1, x_2, \ldots x_n) = \sum_{i=1}^{n} \beta_i x_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \beta_{ij} x_i x_j + \varepsilon_k$$

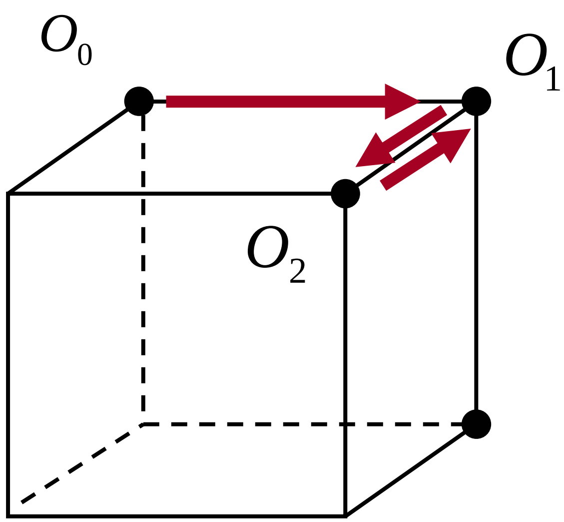
$$\beta_i \sim \mathrm{N}\left(0, \sigma_{ME}{}^2\right) \qquad \beta_{ij} \sim \mathrm{N}\left(0, \sigma_{INT}{}^2\right) \qquad \varepsilon_k \sim \mathrm{N}\left(0, \sigma_{\varepsilon}{}^2\right)$$

$$y_{\max} \equiv \max\left(\sum_{i=1}^{n} \beta_i x_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \beta_{ij} x_i x_j, x_i \in \{-1, +1\} \right)$$

Definition – "Exploiting" Effects

$$x_1^* = \tilde{x}_1 sign\{O_0 - O_1\}$$

$$x_2^* = \tilde{x}_2 sign\{\max(O_0 - O_1) - O_2\}$$



$$\Pr(\beta_1 x_1^* > 0) = ?$$

$$\Pr(\beta_{12} x_1^* x_2^* > 0) = ?$$

What is the likelihood of exploiting a physical effect (main effect or interaction) so that it contributes positively to the final outcome?

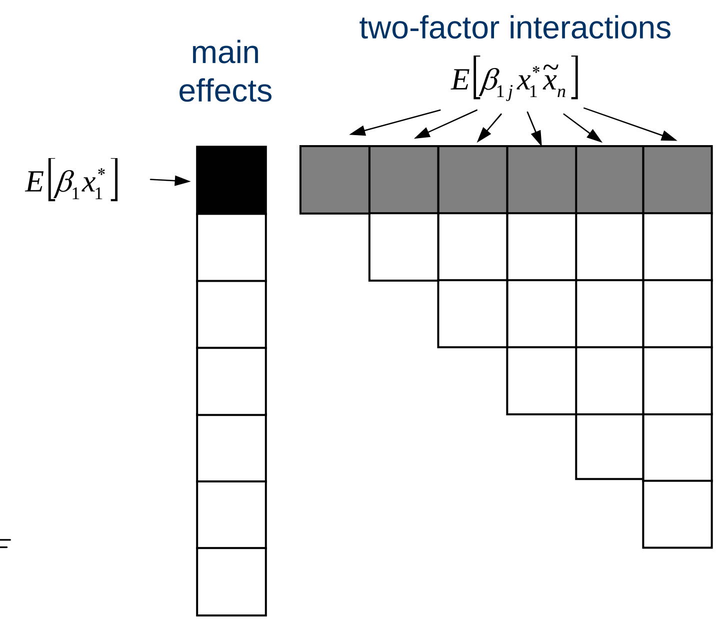
Theory – The First Step

$$E(y(x_1^*,\tilde{x}_2,\ldots,\tilde{x}_n)) = E\left[\beta_1 x_1^*\right] + (n-1)E\left[\beta_{1j} x_1^* \tilde{x}_j\right]$$

$$E\left[\beta_1 x_1^*\right] = \sqrt{\frac{2}{\pi}} \frac{\sigma_{ME}^2}{\sqrt{\sigma_{ME}^2 + (n-1)\sigma_{INT}^2 + \frac{1}{2}\sigma_\varepsilon^2}}$$

$$E\left[\beta_{1j} x_1^* \tilde{x}_j\right] = \sqrt{\frac{2}{\pi}} \frac{\sigma_{INT}^2}{\sqrt{\sigma_{ME}^2 + (n-1)\sigma_{INT}^2 + \frac{1}{2}\sigma_\varepsilon^2}}$$

$$\Pr\left(\beta_1 x_1^* > 0\right) = \frac{1}{2} + \frac{1}{\pi}\sin^{-1}\frac{\sigma_{ME}}{\sqrt{\sigma_{ME}{}^2 + (n-1)\sigma_{INT}{}^2 + \frac{1}{2}\sigma_\varepsilon{}^2}}$$



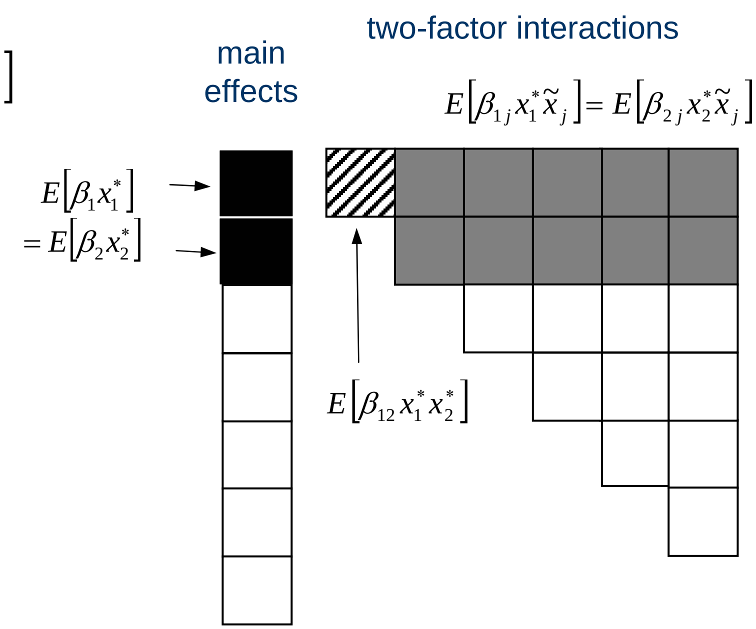
Frey, D. D., and H. Wang, 2006, "Adaptive One-Factor-at-a-Time Experimentation and Expected Value of Improvement", accepted to *Technometrics.*

Theory – Second Step

$$E(y(x_1^*, x_2^*, \tilde{x}_3, \ldots, \tilde{x}_n)) = 2E\left[\beta_1 x_1^*\right] + 2(n-2)E\left[\beta_{1j} x_1^*\right] + E\left[\beta_{12} x_1^* x_2^*\right]$$

$$E\left[\beta_{12} x_1^* x_2^*\right] = \sqrt{\frac{2}{\pi}}\left[\frac{\sigma_{INT}^2}{\sqrt{\sigma_{ME}^2 + (n-1)\sigma_{INT}^2 + \frac{\sigma_\varepsilon^2}{2}}}\right]$$

$$\Pr\left(\beta_{12} x_1^* x_2^* > 0\right) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\frac{\sigma_{INT}}{\sqrt{\sigma_{ME}^2 + (n-2)\sigma_{INT}^2 + \frac{1}{2}\sigma_\varepsilon^2}}$$

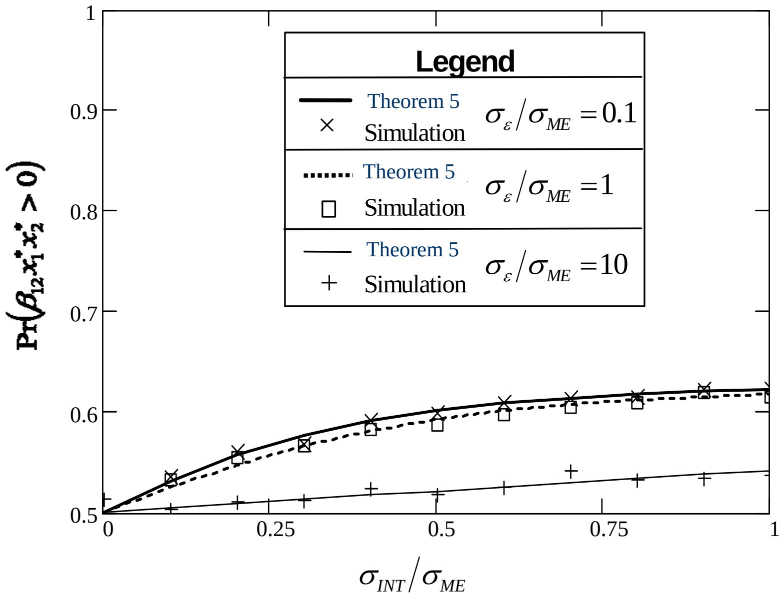


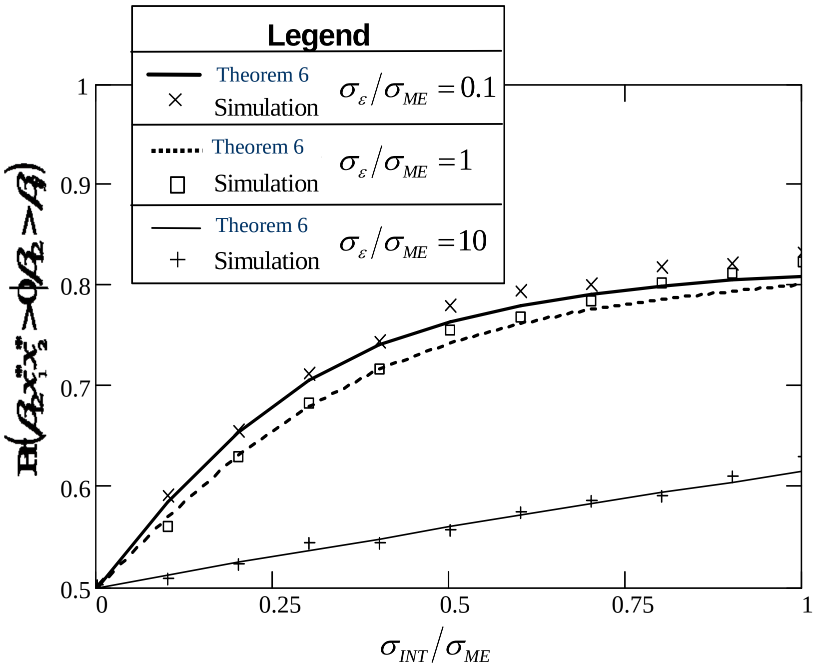
Probability of Exploiting the First Two-Factor Interaction (*n*=7)

$$\Pr\left(\beta_{12}x_1^*x_2^* > 0\right) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\frac{\sigma_{INT}}{\sqrt{\sigma_{ME}^2 + (n-2)\sigma_{INT}^2 + \frac{1}{2}\sigma_\varepsilon^2}}$$

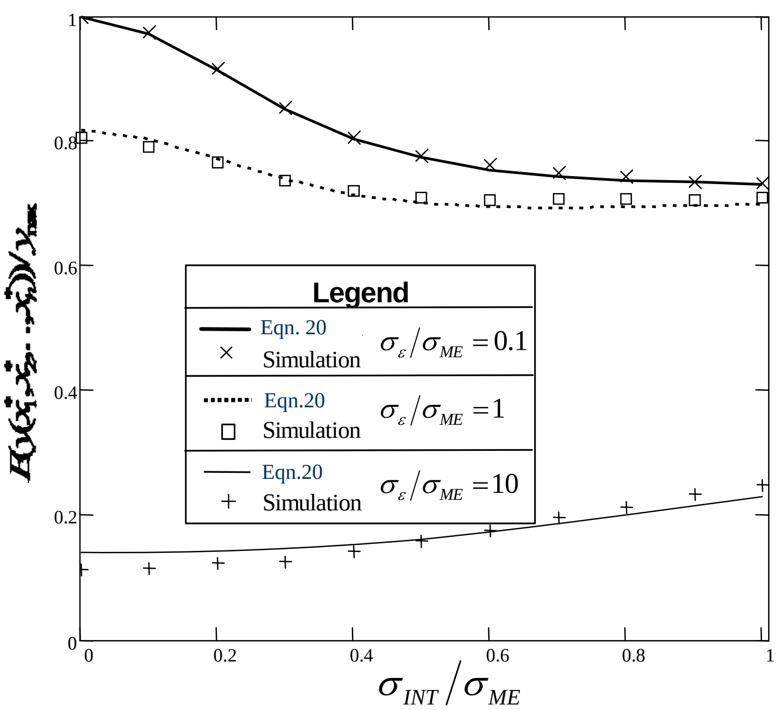
$$\Pr\left(\beta_{12}x_1^*x_2^* > 0 \middle| \beta_{12} > \beta_{ij}\right) >$$

$$\frac{1}{\pi}\binom{n}{2}\int_0^\infty\int_{-x_2}^\infty \frac{\left[erf\left(\frac{1}{\sqrt{2}}\frac{x_1}{\sigma_{INT}}\right)\right]^{\binom{n}{2}-1} e^{\frac{-x_1^2}{2\sigma_{INT}^2}+\frac{-x_2^2}{2\left(\sigma_{ME}^2+(n-2)\sigma_{INT}^2+\frac{1}{2}\sigma_\varepsilon^2\right)}}}{\sigma_{INT}\sqrt{\sigma_{ME}^2 + (n-2)\sigma_{INT}^2 + \frac{1}{2}\sigma_\varepsilon^2}}\,dx_2\,dx_1$$

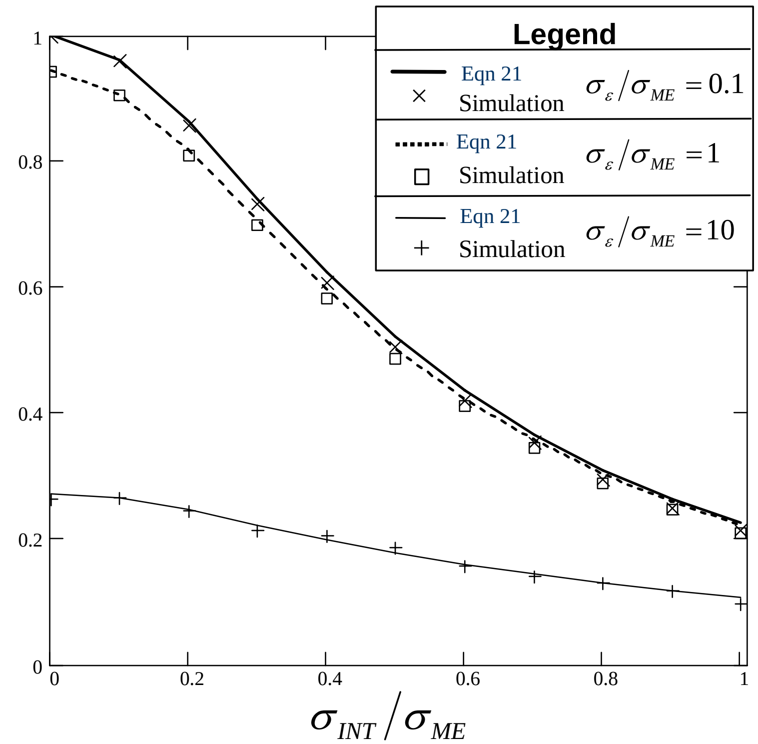




Final Outcome (*n*=7)

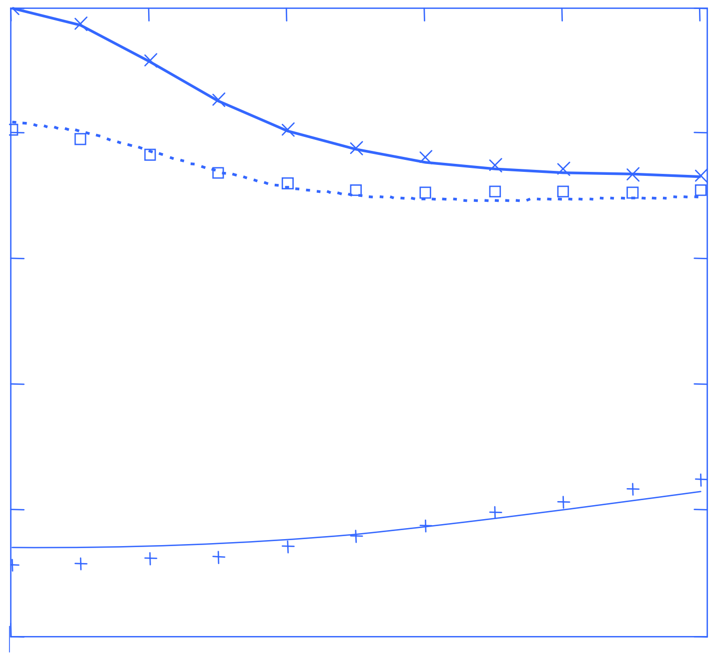


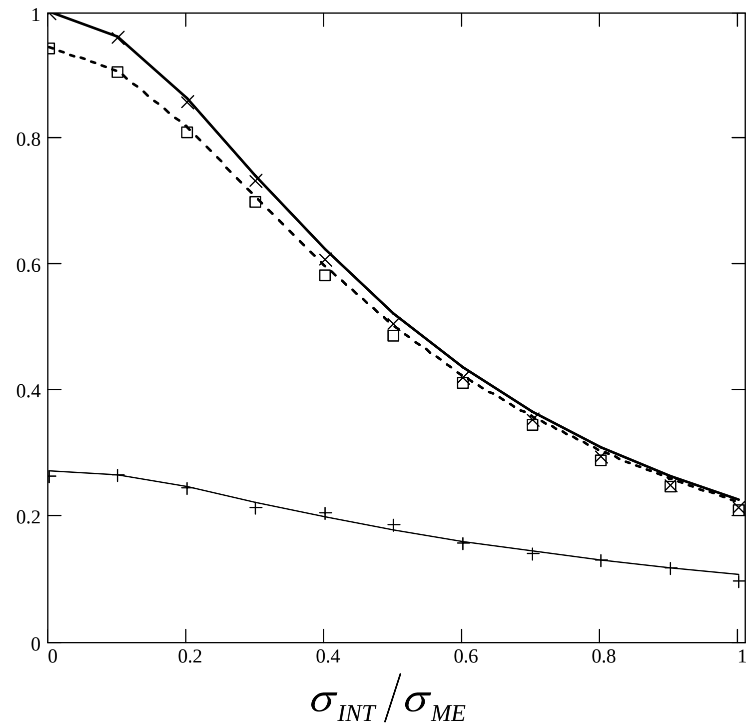
Adaptive OFAT



Resolution *III* Design

Final Outcome (n=7)





Adaptive OFAT

Resolution *III* Design

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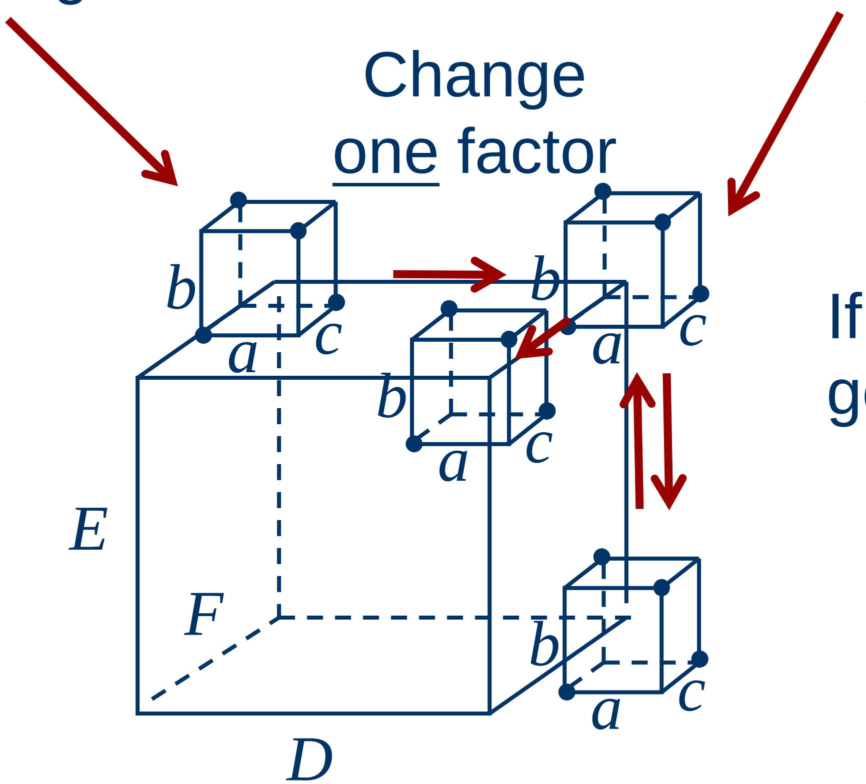
Adaptive OFAT for Robust Design

Run a resolution *III* design on noise factors



Again, run a resolution *III* on noise factors. If there is an improvement, in transmitted variance, retain the change

If the response gets worse, go back to the previous state

Stop after you've changed every factor once

A Hierarchical Probability Model

$$y(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \beta_i x_i + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j>i}}^{n} \beta_{ij} x_i x_j + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j>i}}^{n} \sum_{\substack{k=1 \\ k>j}}^{n} \beta_{ijk} x_i x_j x_k + \varepsilon$$

$$f(\beta_i | \delta_i) = \begin{cases} N(0,1) & \text{if } \delta_\mathrm{i} = 0 \\ N(0, c^2) & \text{if } \delta_\mathrm{i} = 1 \end{cases}$$

effects are normally distributed
two classes – “active” and “inactive”

$$\Pr(\delta_i = 1) = p$$

effect sparsity – a small fraction of all effects are “active”

$$\Pr(\delta_{ij} = 1 | \delta_i, \delta_j) = \begin{cases} p_{00} & \text{if } \delta_\mathrm{i} + \delta_\mathrm{j} = 0 \\ p_{01} & \text{if } \delta_\mathrm{i} + \delta_\mathrm{j} = 1 \\ p_{11} & \text{if } \delta_\mathrm{i} + \delta_\mathrm{j} = 2 \end{cases}$$

inheritance implies

$$p_{11} > p_{01} > p_{00}$$

Adapted from Chipman, H., M. Hamada, and C. F. J. Wu, 2001, “A Bayesian Variable Selection Approach for Analyzing Designed Experiments with Complex Aliasing”, *Technometrics* 39(4)372-381.

Fitting the Model to Data

- Collect published full factorial data on various engineering systems
  - 113 sets collected from published experiments (used for fit)
  - 90 sets collected from Ford Motor Company (a check)
- Lenth method used to sort “active” and “inactive” effects
- Probabilities in the model estimated based on frequencies in the data sets

Li, X. and D. D. Frey, 2005, “Regularities in Data from Factorial Experiments,” accepted to *Complexity*.

Results of Model-Based Evaluation

TABLE 3. Mean percent reduction in transmitted variance achieved by various methods applied to various models.

	$2_{VI}^{6-1} \times 2_{III}^{3-1}$	2^{9-4}		$2_{III}^{6-3} \times 2_{III}^{3-1}$			$aOFAT \times 2_{III}^{3-1}$	
	response modeling	largest noise clear	minimum J-aberration	both analyses	response modeling	classical analysis	p_{guess}= 75%	p_{guess}= 50%
Fitted model	65%	20%	24%	51%	30%	39%	60%	55%
Fitted model, high ε	55%	17%	22%	46%	21%	30%	47%	40%
Sparse	57%	17%	23%	44%	31%	34%	52%	47%
Strong Hierarchy	60%	32%	38%	57%	56%	50%	57%	54%

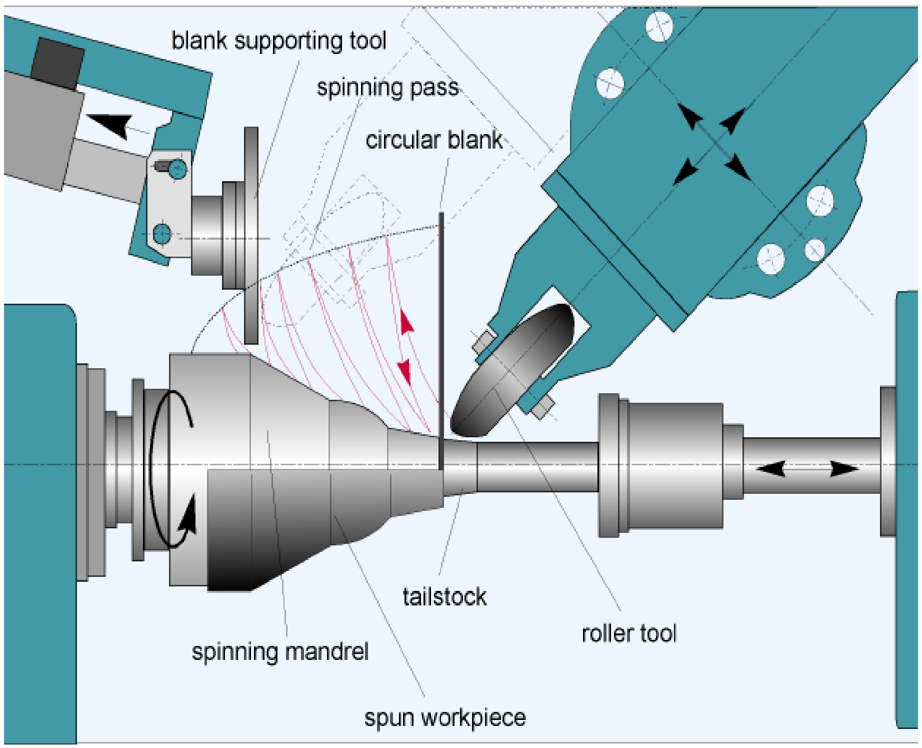
TABLE 4. Inter-quartile range of percent reduction in transmitted variance achieved by various methods applied to various models.

	$2_{VI}^{6-1} \times 2_{III}^{3-1}$	2^{9-4}		$2_{III}^{6-3} \times 2_{III}^{3-1}$			$aOFAT \times 2_{III}^{3-1}$	
	response modeling	largest noise clear	minimum J-aberration	both analyses	response modeling	classical analysis	p_{guess}= 75%	p_{guess}= 50%
Fitted model	46 to 87%	-1 to 53%	3 to 59%	28 to 77%	9 to 65%	14 to 69%	39 to 82%	34 to 77%
Fitted model, high ε	33 to 80%	-2 to 52%	0 to 54%	24 to 72%	4 to 57%	8 to 63%	27 to 73%	17 to 68%
Sparse	32 to 83%	1 to 45%	4 to 48%	20 to 70%	9 to 53%	12 to 58%	28 to 76%	24 to 69%
Strong Hierarchy	31 to 83%	13 to 61%	16 to 66%	31 to 84%	31 to 83%	25 to 74%	31 to 80%	28 to 76%

Frey, D. D., and X. Li, 2006, “Using Hierarchical Probability Models to Evaluate Robust Parameter Design Methods,” currently being revised.

Kunert, J., Auer, C., Erdbrugge, M., and Gobel, R., 2006, "An Experiment to Compare the Combined Array and the Product Array for Robust Parameter Design," accepted to *Journal of Quality Technology*.

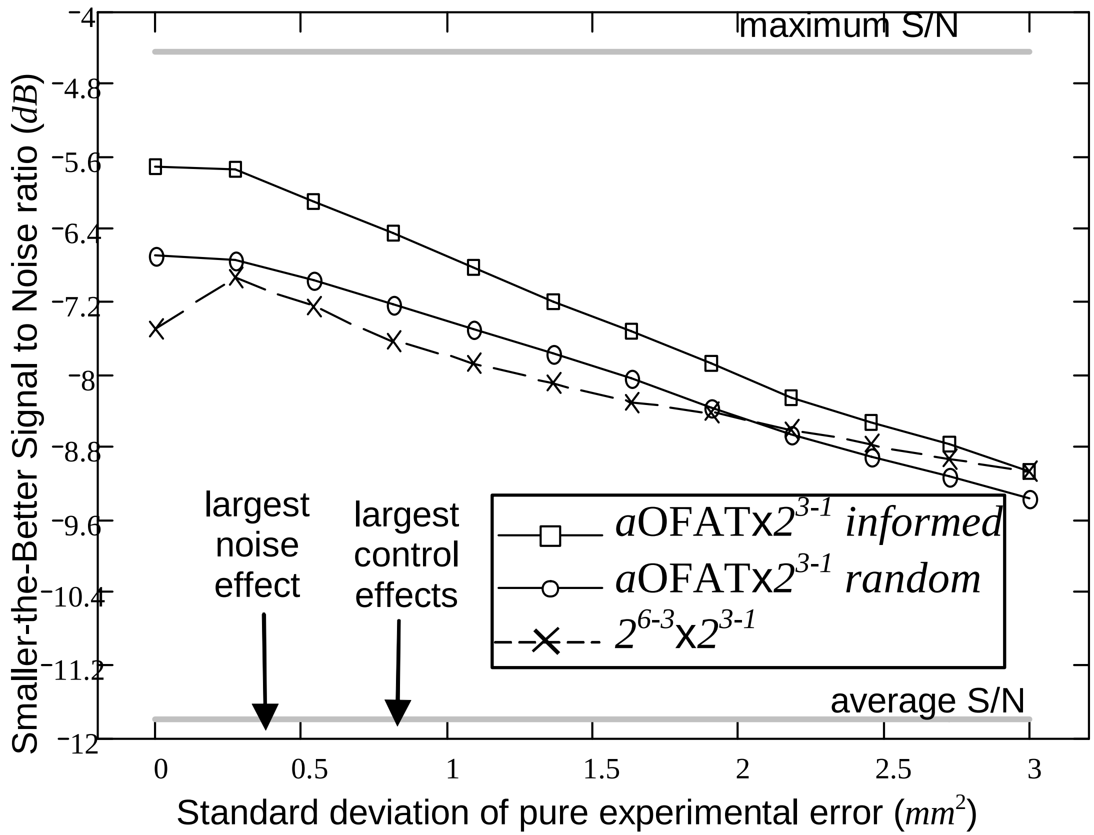
Sheet Metal Spinning -- Model



- Polynomial equation as derived from data by Kunert (2006)
- Control factors A-F
- Noise factors m,n,o

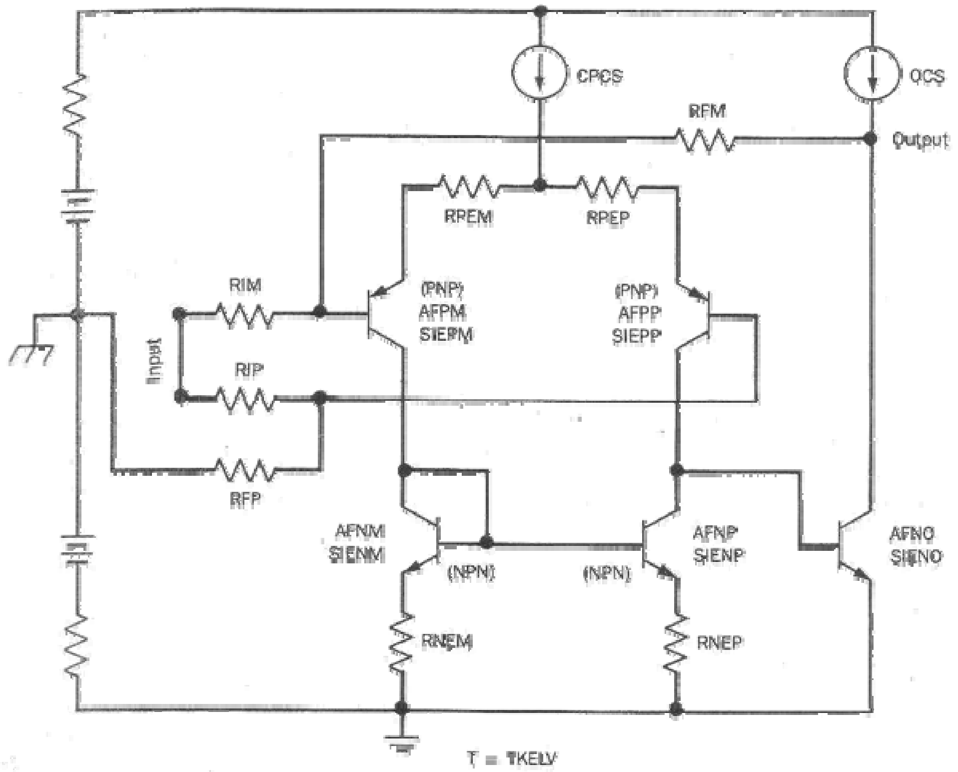
$$A_{20} = 4.1 + 0.8x_F - 0.8x_B + 0.7x_Bx_C + 0.4x_n + 0.2x_Ex_o - 0.2x_Bx_Cx_D$$
$$+ 0.2x_Cx_F + 0.2x_Ax_C - 0.15x_Dx_Ex_o - 0.1x_Cx_Dx_n + 0.1x_Cx_n - 0.1x_Dx_n - 0.1x_Dx_m$$

Sheet Metal Spinning -- Results



Frey, D. D., N. Sudarsanam, and J. B. Persons, 2006, "An Adaptive One-Factor-At-A-Time Method for Robust Parameter Design: Comparison with Crossed Arrays via Case Studies," accepted to *ASME Design Engineering Technical Conferences*.

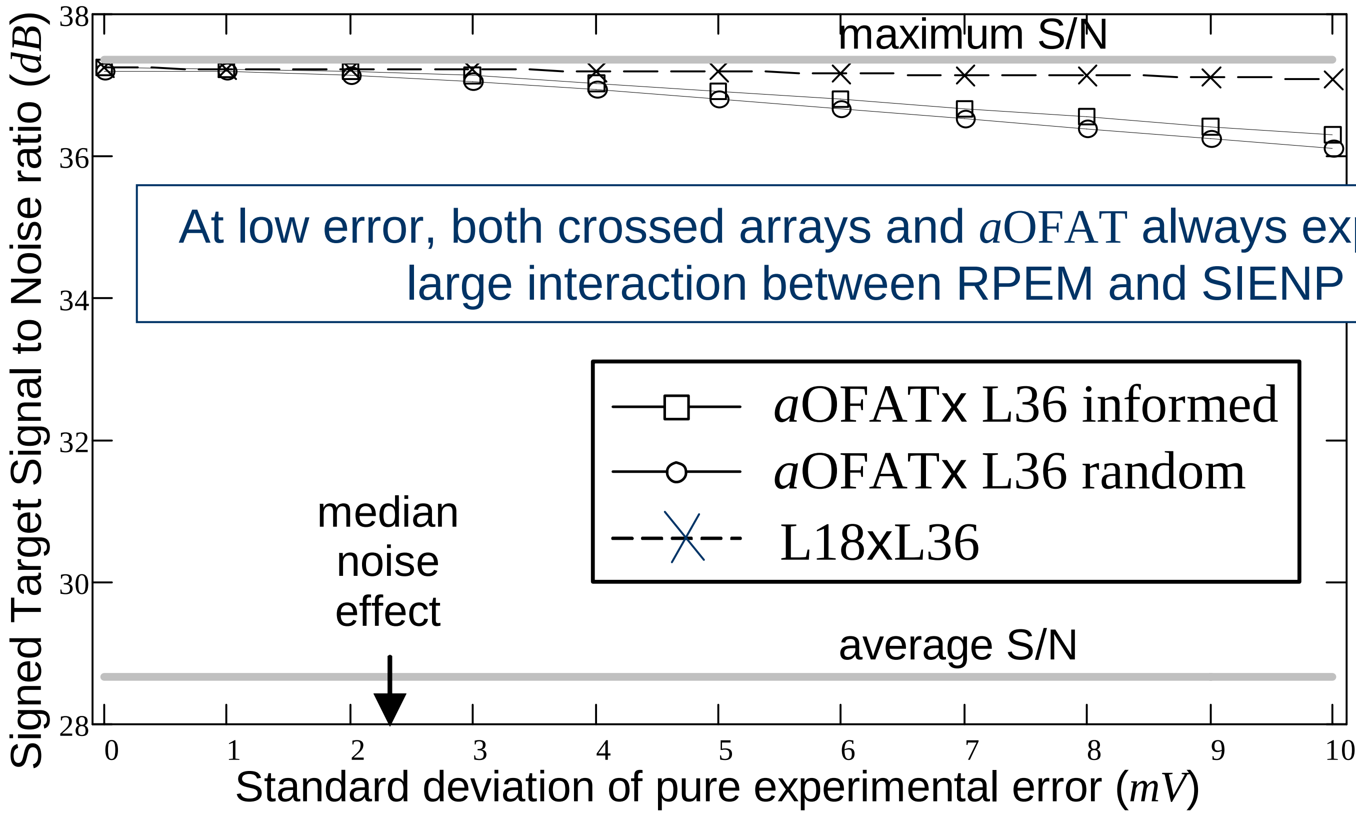
Operational Amplifier -- Model



- Ebers-Moll model of the transistors
- Equations implemented in closed form
- Matlab solver
- Same 21 noise factors as presented in Phadke (1989)

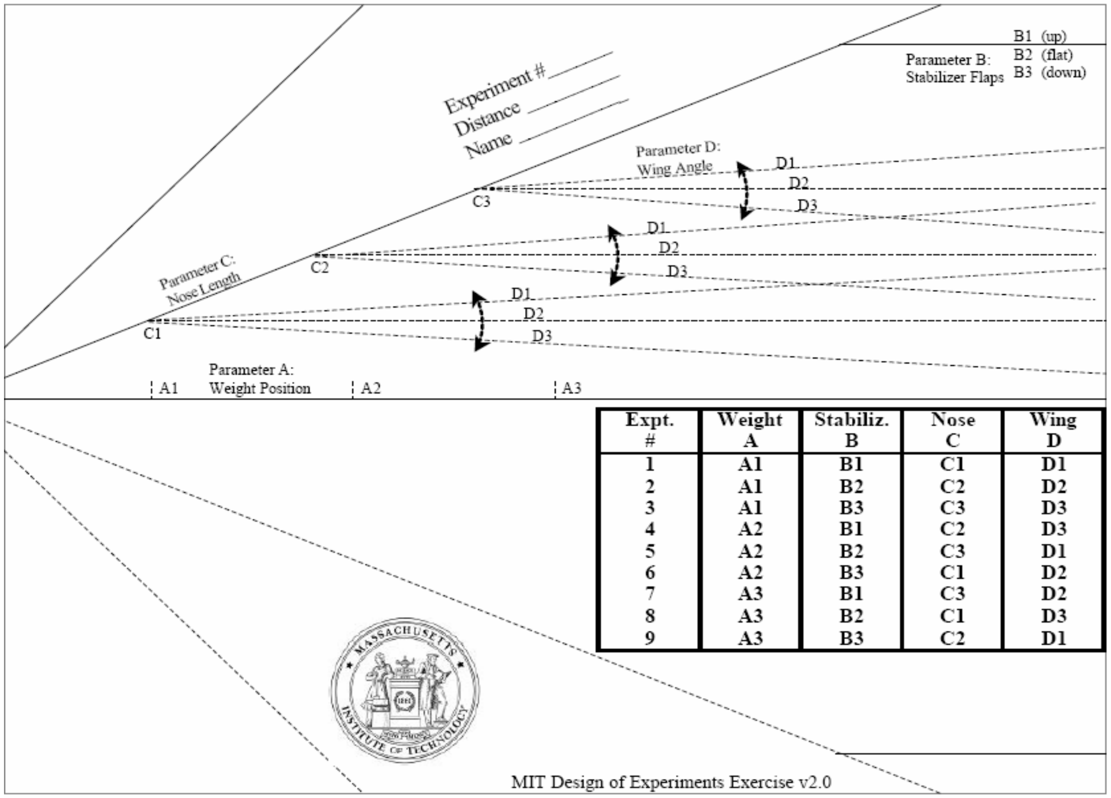
Phadke, Madhav S., 1989, *Quality Engineering Using Robust Design*, Prentice Hall, Englewood Cliffs, NJ.

Operational Amplifier -- Results



At low error, both crossed arrays and aOFAT always exploited the large interaction between RPEM and SIENP

Paper Airplane – “Model”

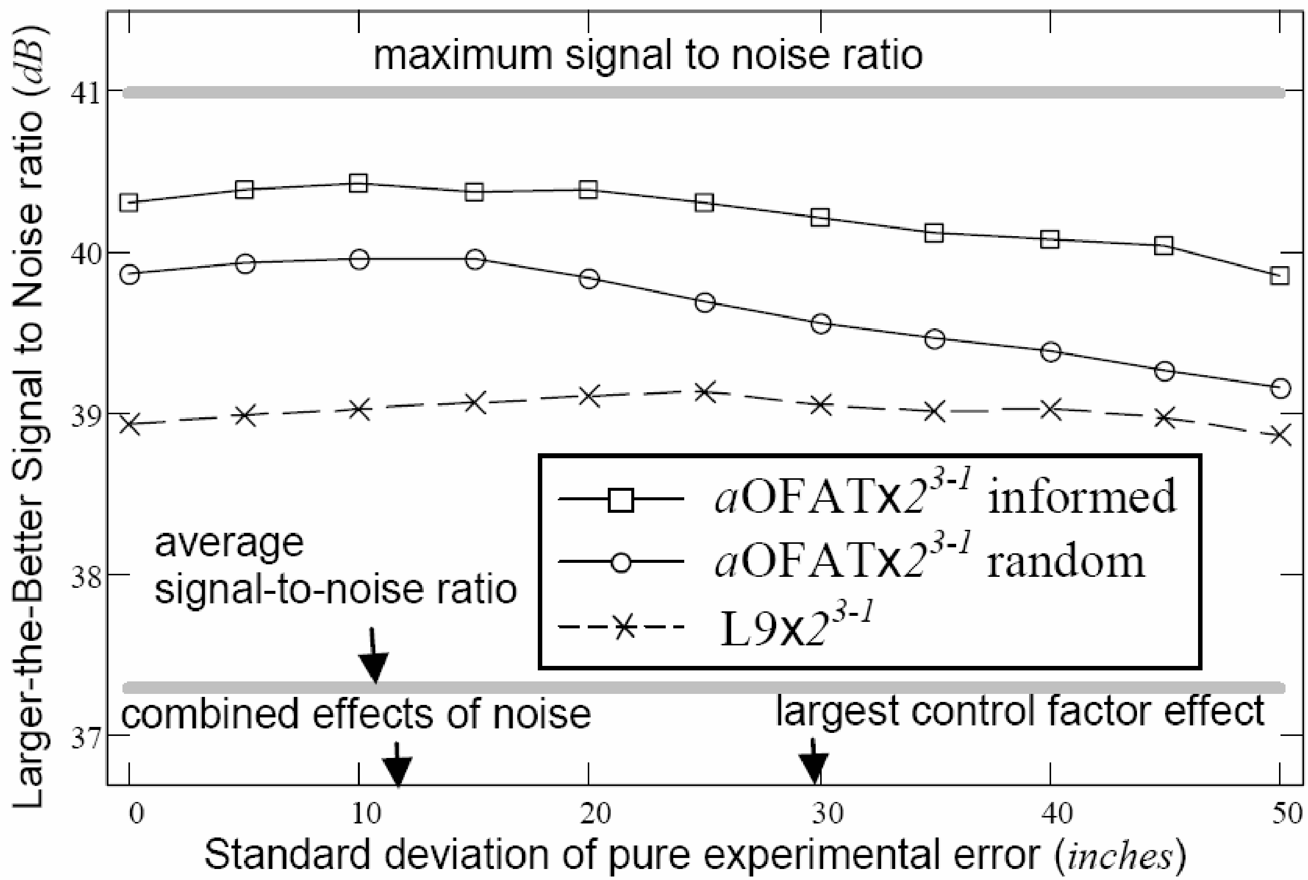


Expt. #	Weight A	Stabiliz. B	Nose C	Wing D
1	A1	B1	C1	D1
2	A1	B2	C2	D2
3	A1	B3	C3	D3
4	A2	B1	C2	D3
5	A2	B2	C3	D1
6	A2	B3	C1	D2
7	A3	B1	C3	D2
8	A3	B2	C1	D3
9	A3	B3	C2	D1

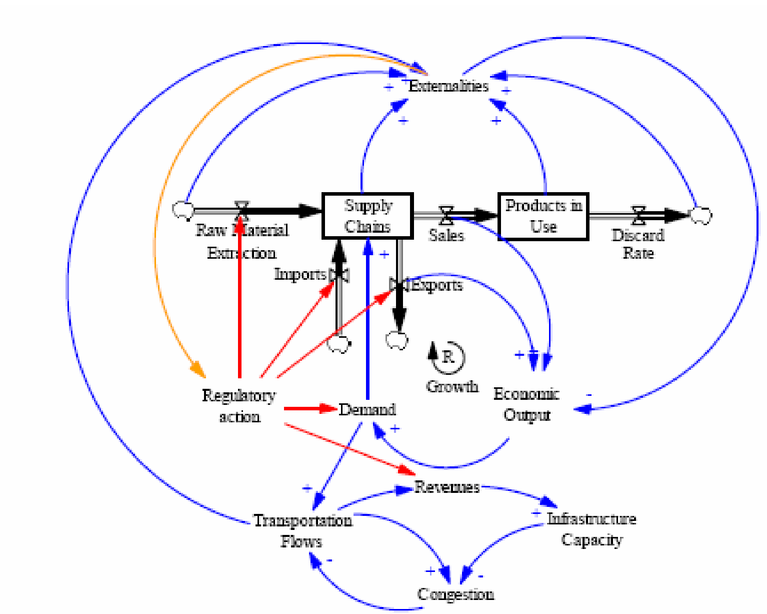
- The data itself as tabulated from a $3^4 x 2^{3-1}$
- Experimental error added using a random number generator

For *aOFAT* random, probabilities are,
~98% to exploit the largest control by noise interaction
>80% to exploit the largest control by control interaction
>60% to exploit the largest control by control by noise interaction

Paper Airplane – Results



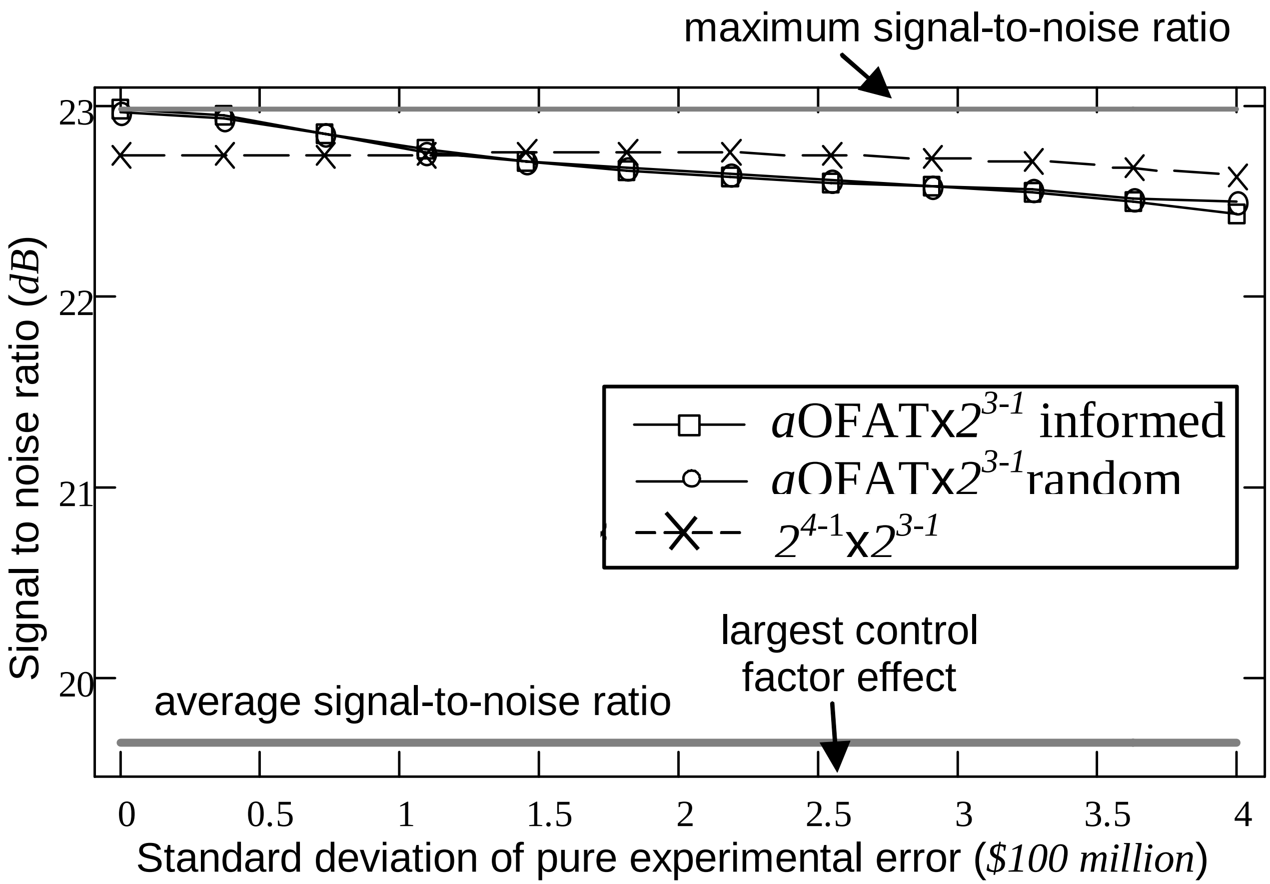
Freight Transportation – Model



- “System dynamics” model in AnyLogic©, version 5.2
- Control variables = gas tax, truck weight limit, product tax and carbon emission tax levels
- Noise variables = fuel efficiency of the truck, the landfill price, and truck emissions
- Multiple output variables rolled into a single objective function

Glazner, C., and Sgouridis, S., 2005, “Optimizing Freight Transportation Policies for Sustainability”, MIT Sloan School of Management. http://www.xjtek.com/files/papers/freighttransportation2005.pdf

Freight Transportation – Results



Summary of Case Study Results

		Method used		
		Fractional array $\times 2_{III}^{k-p}$	$aOFAT \times 2_{III}^{k-p}$	
			informed	*random*
sheet metal spinning $S/N_{max} = 7.4$	Low ε	4.3	6.1	5.1
	High ε	2.7	2.7	2.4
op amp $S/N_{max} = 8.6$	Low ε	8.5	8.5	8.4
	High ε	8.4	7.6	7.5
paper airplane $S/N_{max} = 3.7$	Low ε	1.6	3.0	2.5
	High ε	1.5	2.5	1.9
freight transport $S/N_{max} = 3.3$	Low ε	3.1	3.3	3.3
	High ε	2.9	2.8	2.8
Mean of four cases	Low ε	4.4	5.2	4.8
	High ε	3.9	3.9	3.7

Conclusions: *a*OFAT applied to RPD

<u>Adaptive OFAT</u>

- As long as experimental error is not much larger than main effects
- Exploits main effects
- Exploits 2-factor interactions, especially the largest ones
- Provides about 80% of the maximum possible improvement on average

<u>Adaptive OFAT X Noise Array</u>

- As long as experimental error is not much larger than relevant effects
- Exploits CXN interactions
- Exploits CXCXN interactions, especially the largest ones
- Provides about 80% of the maximum possible improvement on average
- Works even better with an informed starting point

- Generally better results than alternatives with similar resource demands

Acknowledgements

- *NSF CAREER grant* #0448972
- Ford / MIT Alliance
- Many excellent students

Questions?

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Friedman and Savage on Adaptation

"The factorial design is ideally suited for experiments whose purpose is to map a function in a pre-assigned range."

"…however, the factorial design has certain deficiencies … It devotes observations to exploring regions that may be of no interest."

"…These deficiencies of the factorial design suggest that an efficient design for the present purpose ought to be sequential; that is, ought to adjust the experimental program at each stage in light of the results of prior stages."

Friedman, Milton, and L. J. Savage, 1947, "Planning Experiments Seeking Maxima", in *Techniques of Statistical Analysis*, pp. 365-372.

- TITLE: ADAPTIVE EXPERIMENTATION, EXPECTED VALUE OF IMPROVEMENT, AND ROBUST DESIGN

  ABSTRACT:
  This seminar will present research into adaptive experimentation as a means for making improvements in design of engineering systems. A simple method for experimentation is described entitled adaptive one-factor-at-a-time. A mathematical model is proposed and theorems are presented concerning the expected value of the improvement provided and the probability that factor effects will be exploited. It is shown that adaptive one-factor-at-a-time provides a large fraction of the potential improvements if experimental error is not large compared to the main effects and that this degree of improvement is more than that provided by resolution III fractional factorial designs if interactions are not small compared to main effects. The theorems also establish that the method exploits two-factor interactions when they are large and exploits main effects if interactions are small. A case study on design of electric powered aircraft supports these results. These results are then extended to robust parameter design. A method is proposed for evaluating the effectiveness of robust parameter design methods. The method employs a hierarchical probability model to express assumptions about effect sparsity, hierarchy, and inheritance. Using this approach, it is shown that adaptive one-factor-a-a-time crossed with a resolution III array performs well in comparison to alternative methods in terms of the improvements attained and the costs incurred.