Skill, ROC curves, and optimal forecast combination

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	Probability forecast	**Medical diagnosis**
raw	$\widetilde{X} \in [0,1]$	$\widetilde{X} \in \Re$
decision	$X = I(\widetilde{X} > \gamma)$	$X = I(\widetilde{X} > x_c)$
	PoP	*Appendicitis*
	50% chance rain	white blood count

Loss (k_{YX}) matrix

	$Y = 1$	$Y = 0$
$X = 1$	0	γ
$X = 0$	$1 - \gamma$	0

Optimal Naive Forecast

$$\theta_y = P(Y = 1)$$

$$ONF = I(\theta_y > \gamma) \equiv 0$$

Model

$$P(y,x) = \theta_{yx} = \theta_{y|x}\theta_x$$

$$\begin{aligned} L(y,x|\theta) &= \theta_{1|1}^{yx}(1-\theta_{1|1})^{(1-y)x} \times \\ & \theta_{0|0}^{(1-y)(1-x)}(1-\theta_{0|0})^{y(1-x)} \times \\ & \theta_x^{x}(1-\theta_x)^{1-x} \end{aligned}$$

$$\begin{aligned} p(\theta_{1|1},\ \theta_{0|0}\ ,\theta_x) &= p(\theta_{1|1})p(\theta_{0|0})p(\theta_x) \\ &\sim \text{``}Be_{\theta_{1|1}}(0,0) \times Be_{\theta_{0|0}}(0,0) \times Be_{\theta_x}(0,0)\text{''} \end{aligned}$$

$$p(\theta_{1|1}|y,x) \sim Be(n_{11}, n_{01})$$

$$p(\theta_{0|0}|y,x) \sim Be(n_{00}, n_{10})$$

$$p(\theta_x|y,x) \sim Be(n_{.1}, n_{.0})$$

$$E(\theta_{1|1}|y,x) = \frac{n_{11}}{n_{11} + n_{01}}$$

$$E(\theta_{0|0}|y,x) = \frac{n_{00}}{n_{00} + n_{10}}$$

$$E(\theta_x|y,x) = \frac{n_{11} + n_{01}}{n_{..}}$$

Skill score

Loss matrix

	$Y=1$	$Y=0$
$X=1$	0	γ
$X=0$	$1-\gamma$	0

$$
\begin{aligned}
E(k^{ONF}) &= \theta_y(1-\gamma) \\
&= \Big(\theta_{1|1}\theta_x + (1-\theta_{0|0})(1-\theta_x)\Big)(1-\gamma)
\end{aligned}
$$

$$
E(k^{E}) = \theta_{1|1}\theta_x\gamma + (1-\theta_{0|0})(1-\theta_x)(1-\gamma)
$$

$$K_\gamma = \frac{E\left(k^{ONF}\right) - E\left(k^{E}\right)}{E\left(k^{ONF}\right)}$$

$$= \frac{\theta_x(\theta_{1|1} - \gamma)}{\theta_y(1-\gamma)}$$

Skill : $K_{1/2} > 0$ **Value** : $K_\gamma > 0$

$$P(K_\gamma > 0|y,x) \longleftrightarrow P(\theta_{1|1} > \gamma|y,x)$$

$$E(K_\gamma | y, x) = \frac{n_{11}(1-\gamma) - n_{01}\gamma}{(n_{11}+n_{10})(1-\gamma)}$$

$$E(K_{1/2} | y, x) = \frac{n_{11} - n_{01}}{n_{11}+n_{10}}$$

Example 1: Mammograms

	$Y=1$	$Y=0$
$X=1$	7	70
$X=0$	1	922

$$Accuracy\left(X^{ONF}\right) \;=\; 97\% \qquad Accuracy\left(X^{E}\right) = 93\%$$

$$\widehat{K}_{1/2} \;=\; -7.9 \qquad \left(\gamma < \frac{1}{11} \rightarrow K_\gamma > 0\right)$$

$$P(K_{1/2} > 0|y,x) = P(\theta_{1|1} > 1/2|y,x) \approx 10^{-15}$$

Continuous forecasts skill score

$$F_j = P(Y = j, X \leq x_c)$$

$$K_{\gamma,x_c} = \frac{(1-\gamma)F_1 - \gamma F_0}{\theta_y(1-\gamma)} I_0 + \frac{-(1-\gamma)F_1 + \gamma F_0}{(1-\theta_y)\gamma}(1 - I_0)$$

$$(I_0 = I(\theta_y \leq \gamma))$$

Optimal decision point

$$x_* = \left\{ x : \frac{f(x|Y=1)}{f(x|Y=0)} = \frac{\gamma}{1-\gamma} \frac{1-\theta_y}{\theta_y} \right\}$$

$\equiv$ **Bayesian classifier**

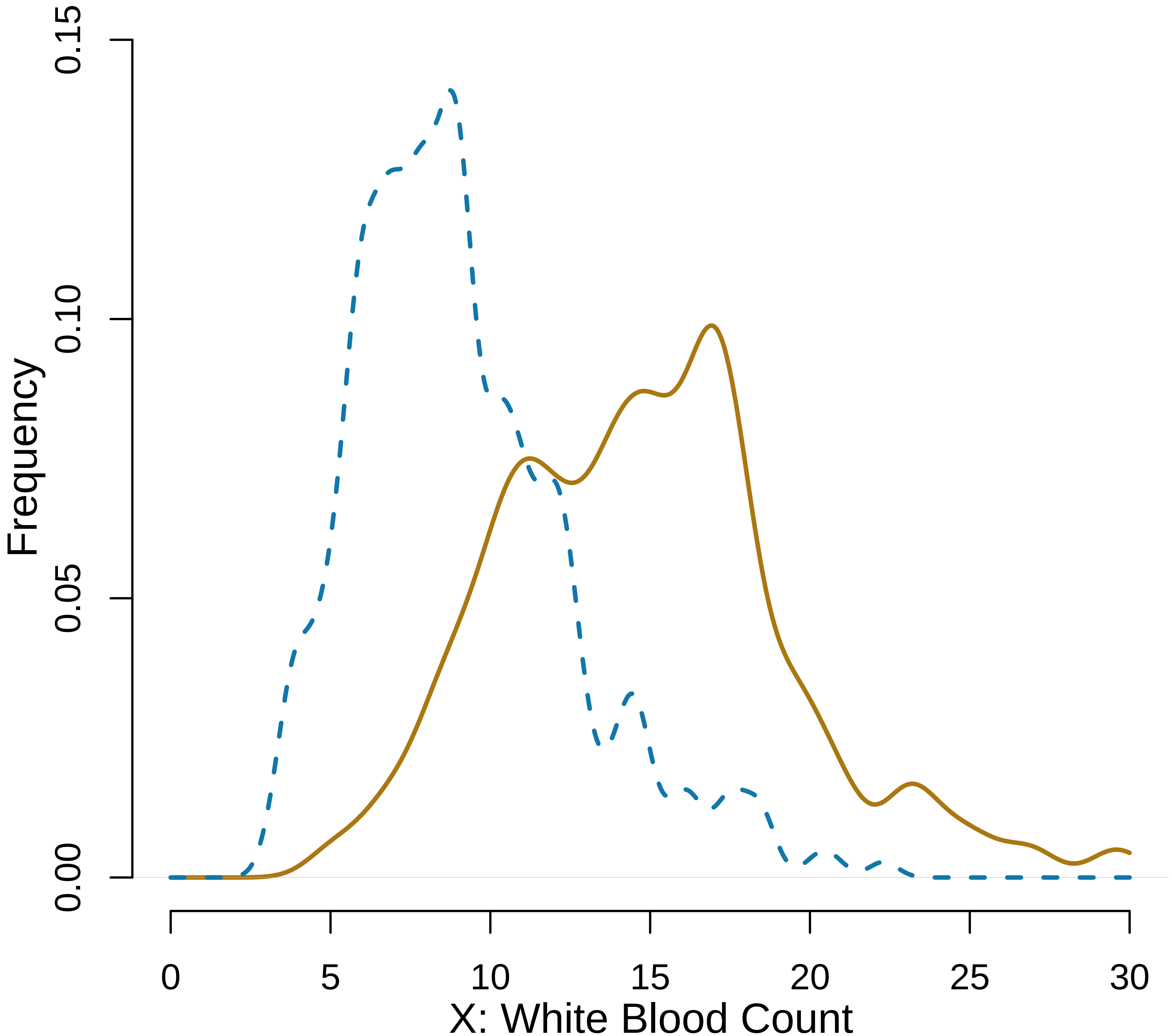
$\equiv$ **ROC curve decision**

Example 2: Appendicitis

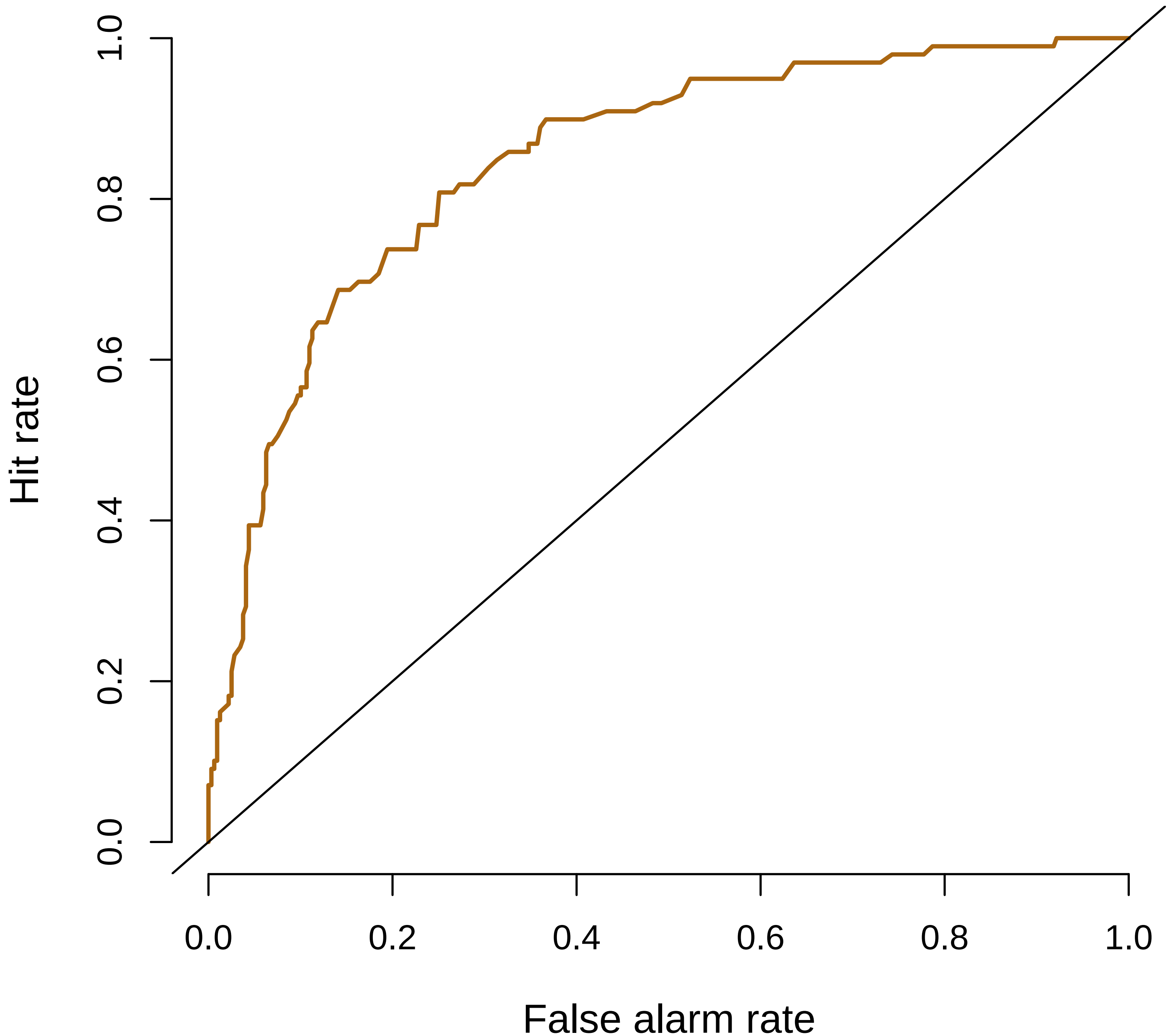
$$X = I(\text{White Blood Count } > x_c)$$

$$= 1 \qquad \text{Cut open}$$

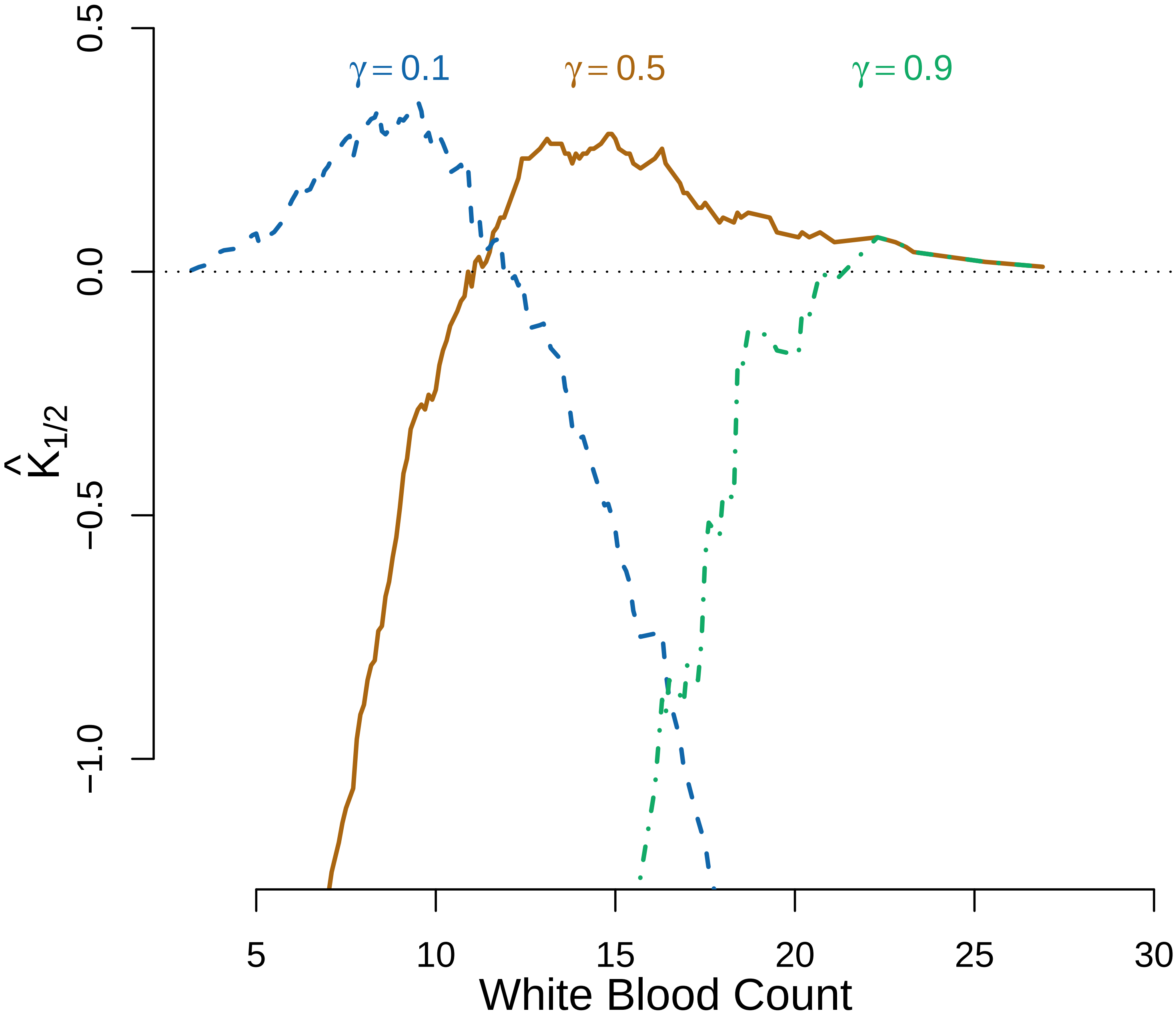
$$= 0 \qquad \text{Send home}$$



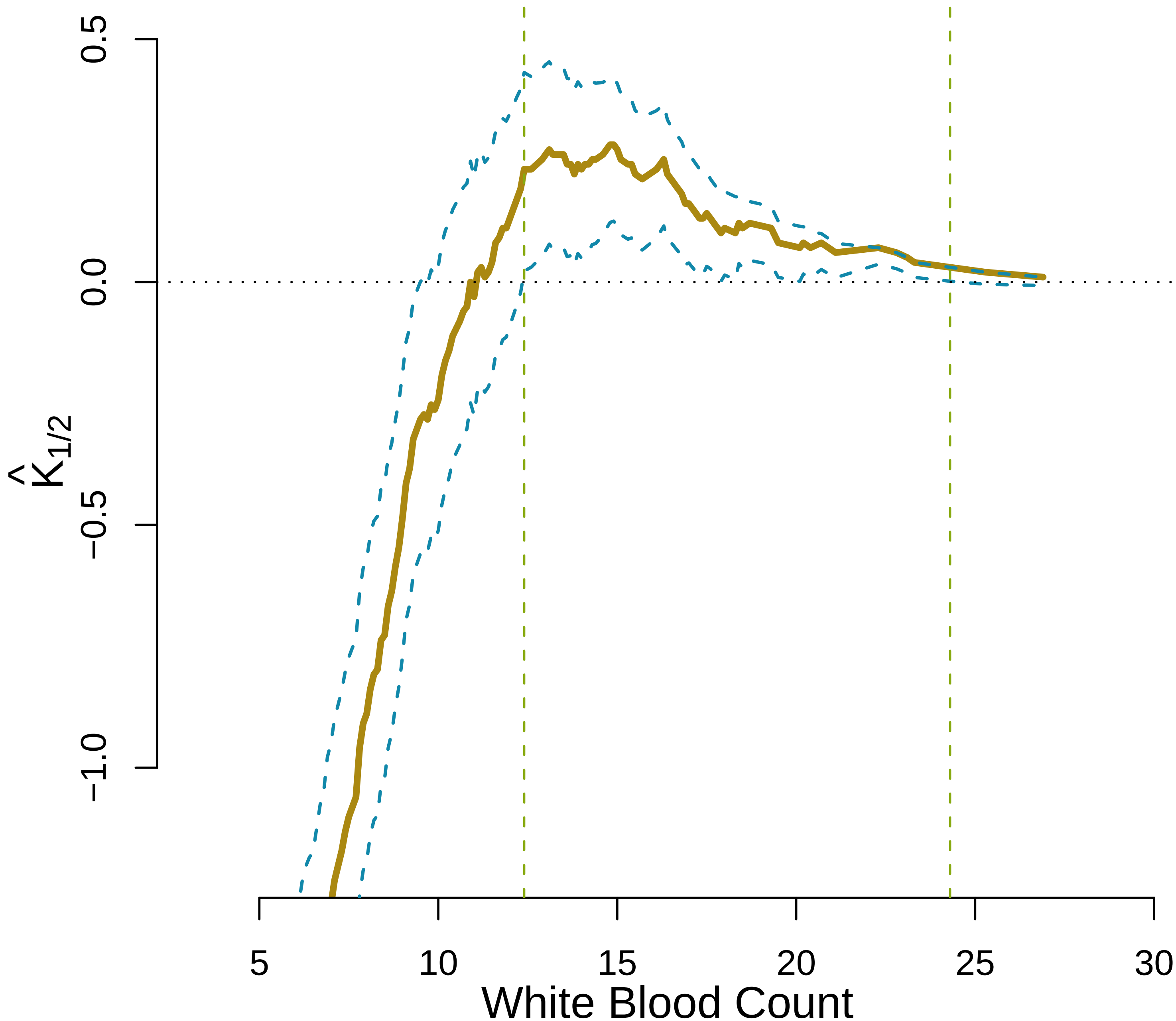
Frequency
0.00
0.05
0.10
0.15
0
5
10
15
20
25
30
X: White Blood Count



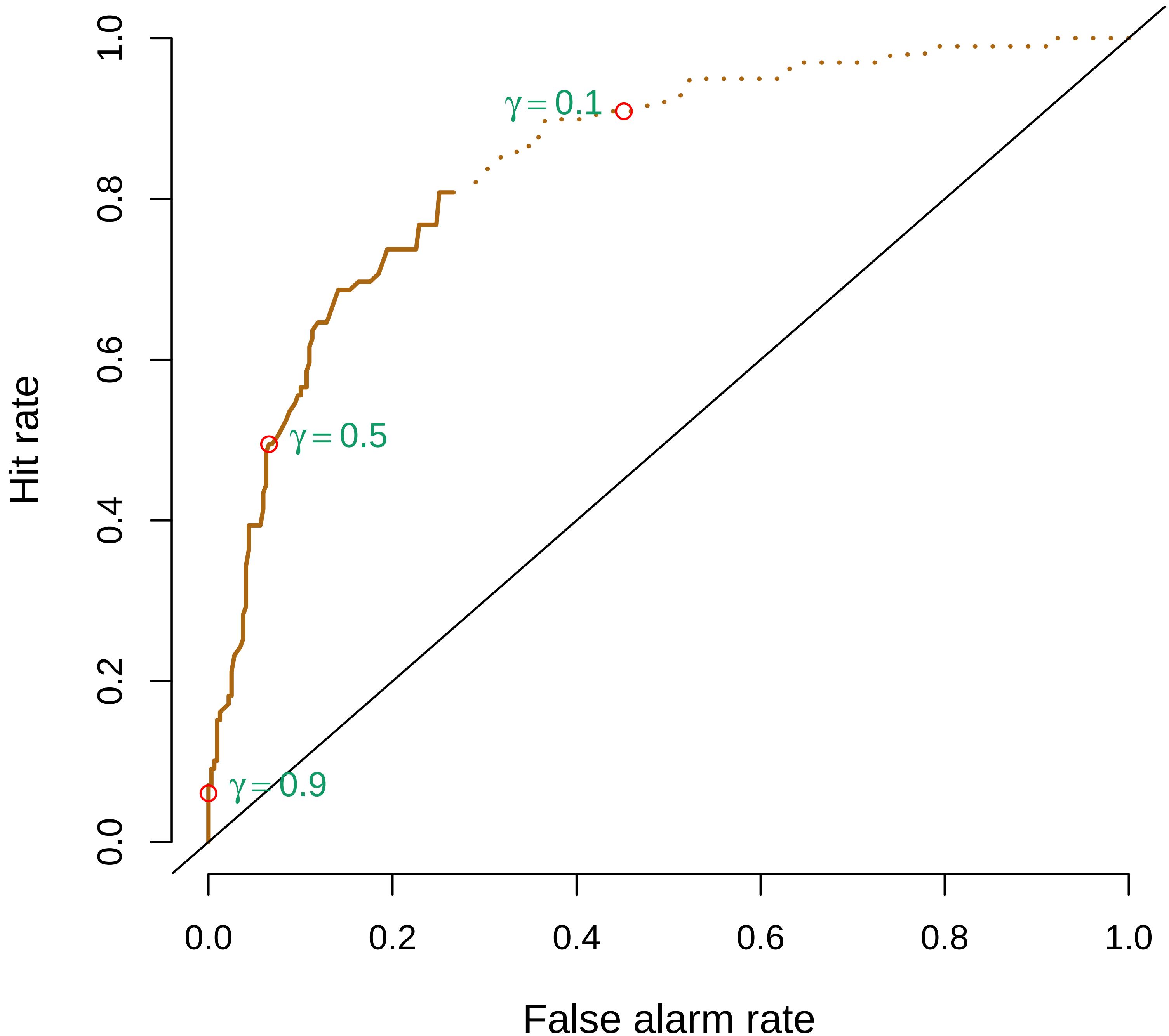
Hit rate
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.2
0.4
0.6
0.8
1.0
False alarm rate

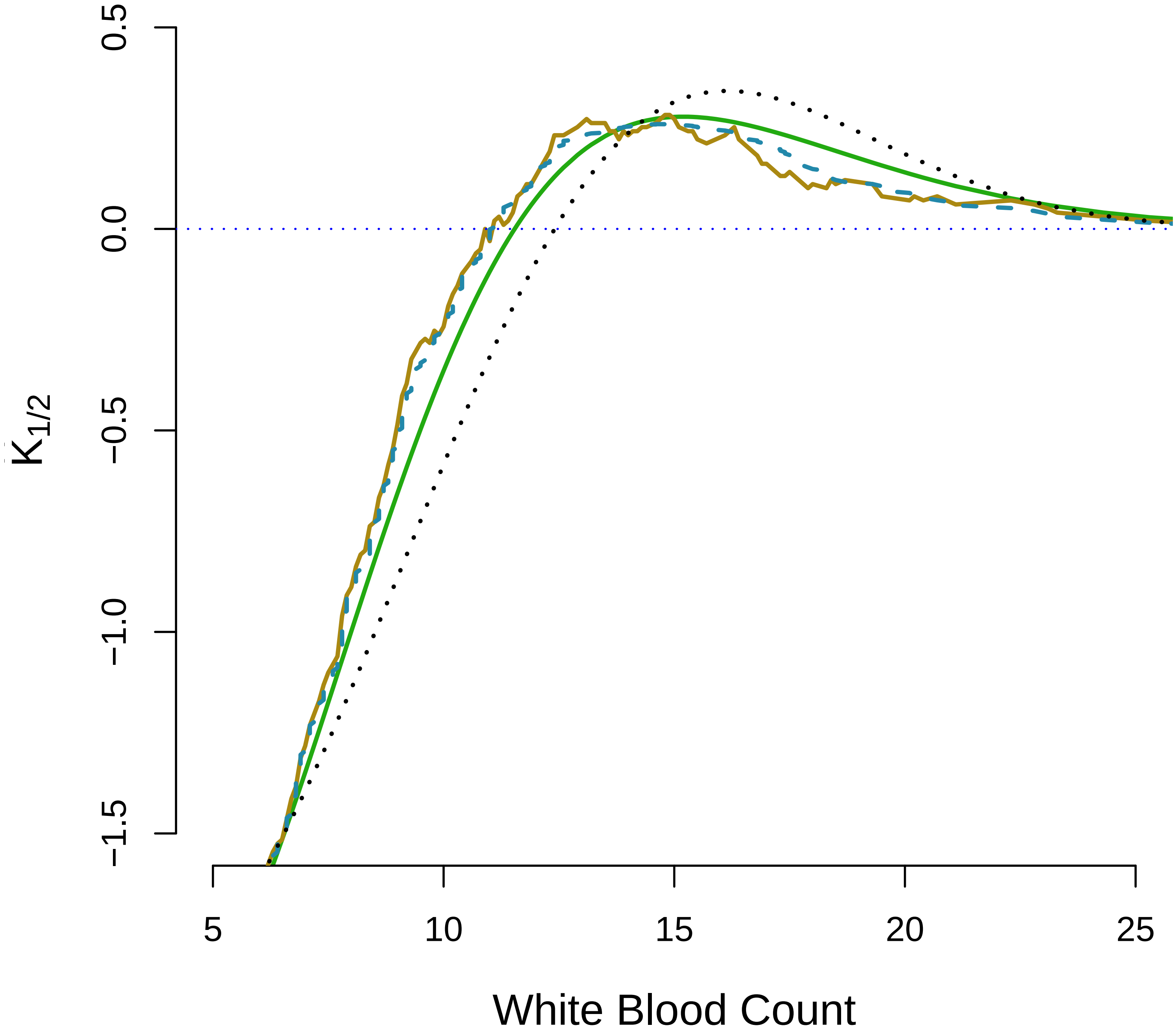


γ = 0.1
γ = 0.5
γ = 0.9
0.5
0.0
−0.5
−1.0
$\hat{K}_{1/2}$
5
10
15
20
25
30
White Blood Count



0.5
0.0
−0.5
−1.0
$\hat{K}_{1/2}$
5
10
15
20
25
30
White Blood Count


γ = 0.1
γ = 0.5
γ = 0.9
Hit rate
0.0
0.2
0.4
0.6
0.8
1.0
False alarm rate
0.0
0.2
0.4
0.6
0.8
1.0



0.5
0.0
−0.5
−1.0
−1.5
$K_{1/2}$
5
10
15
20
25
White Blood Count

Example 3: Combine Two Forecasts: $A, B \in [0, 1]$

$$X = I(wA + (1 - w)B > \gamma)$$

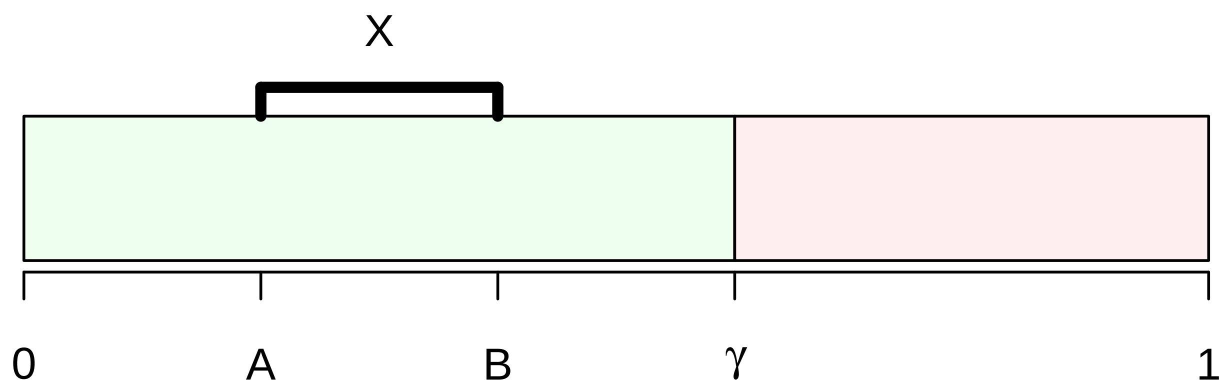
$$(A = \text{Humans}, \quad B = \text{MOS})$$

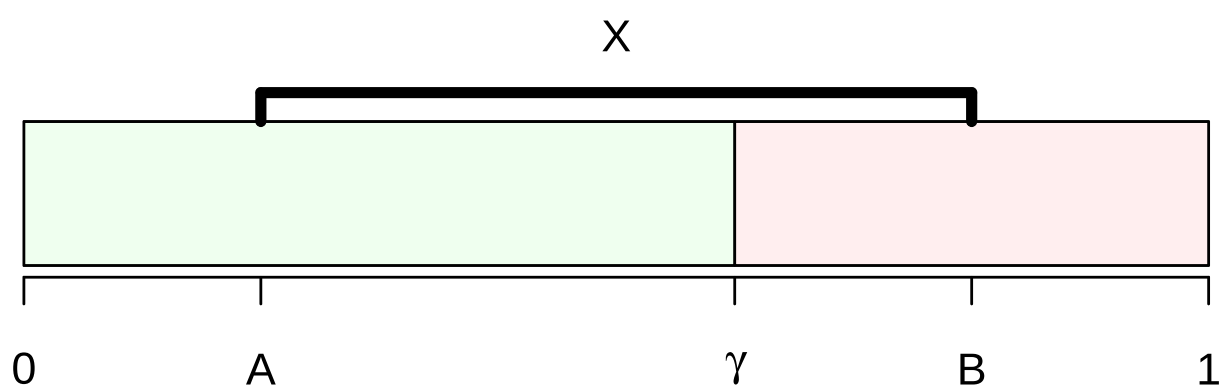
If

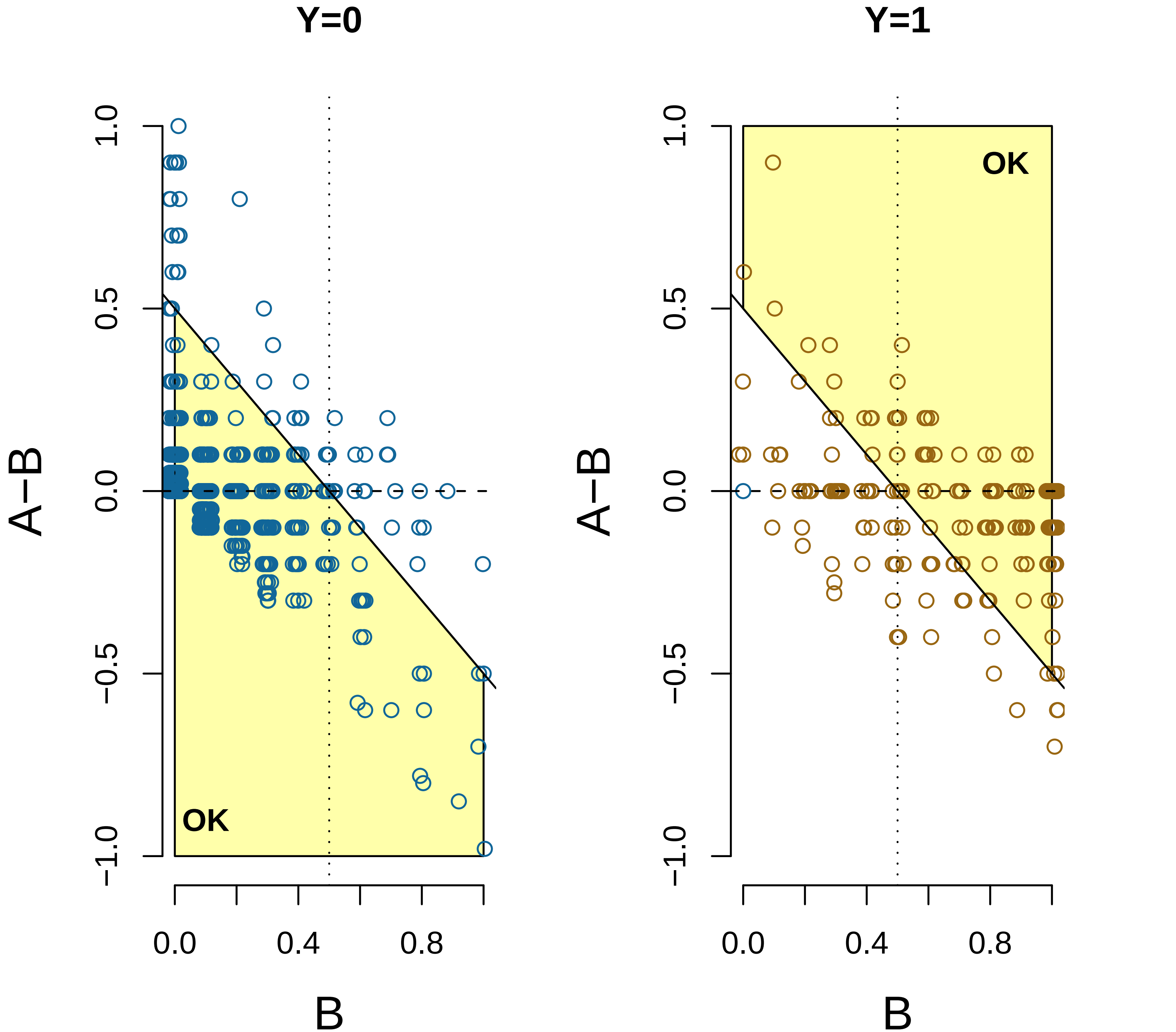
$$A, B \in \{0, 1\}, \qquad K_B > K_A$$

Then

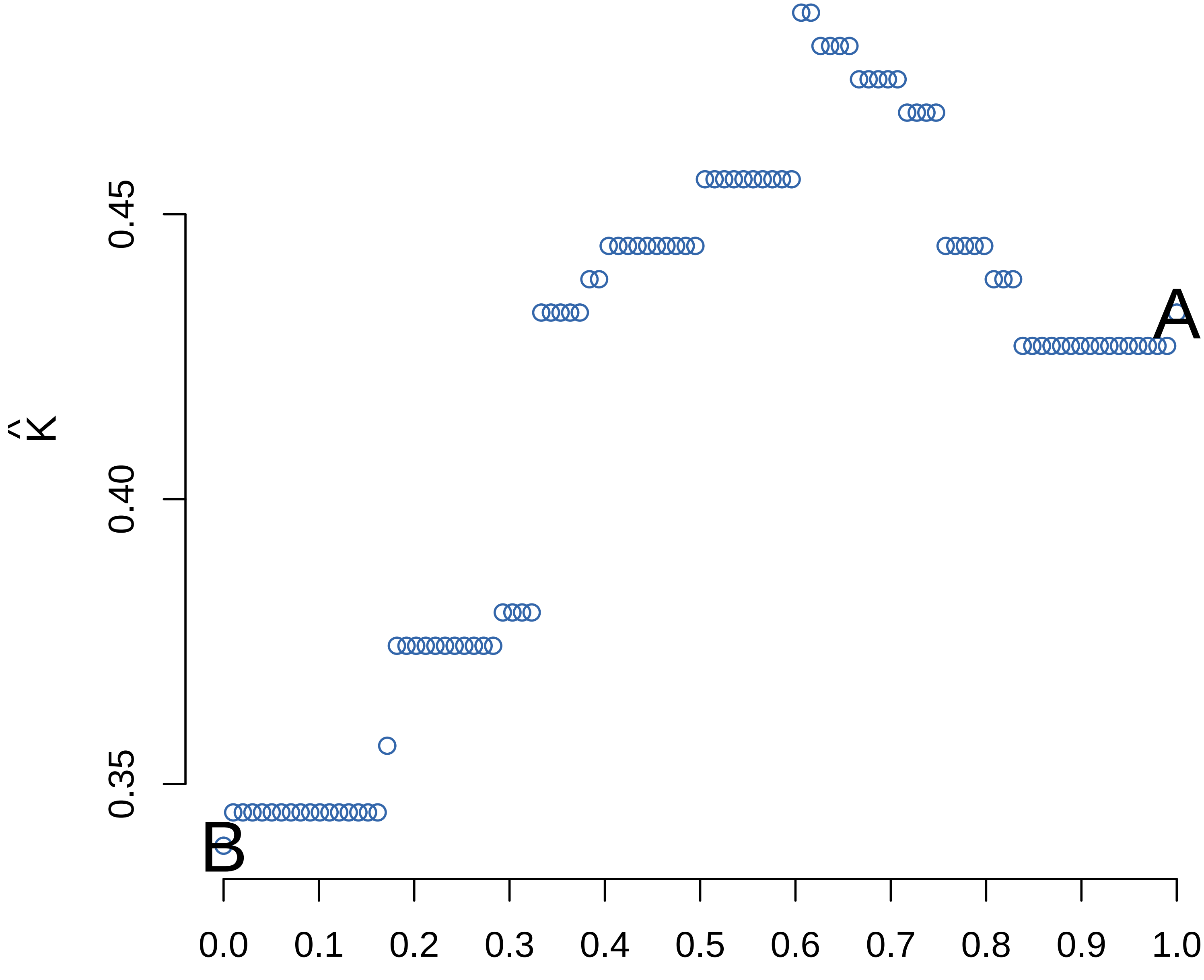
$$w = 0, \qquad \widetilde{X} \equiv B$$


X
0
A
B
γ
1

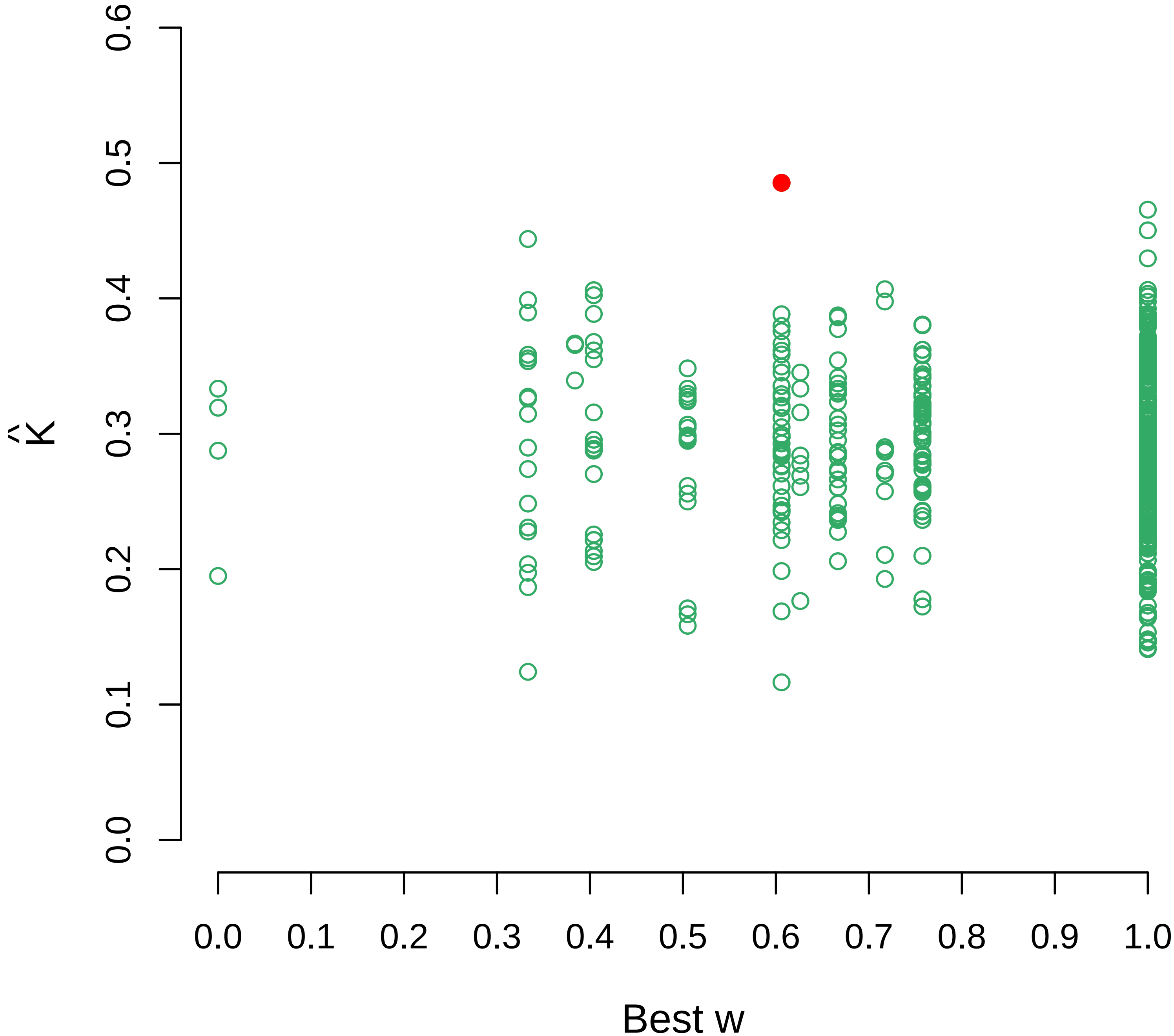

X
0
A
γ
B
1



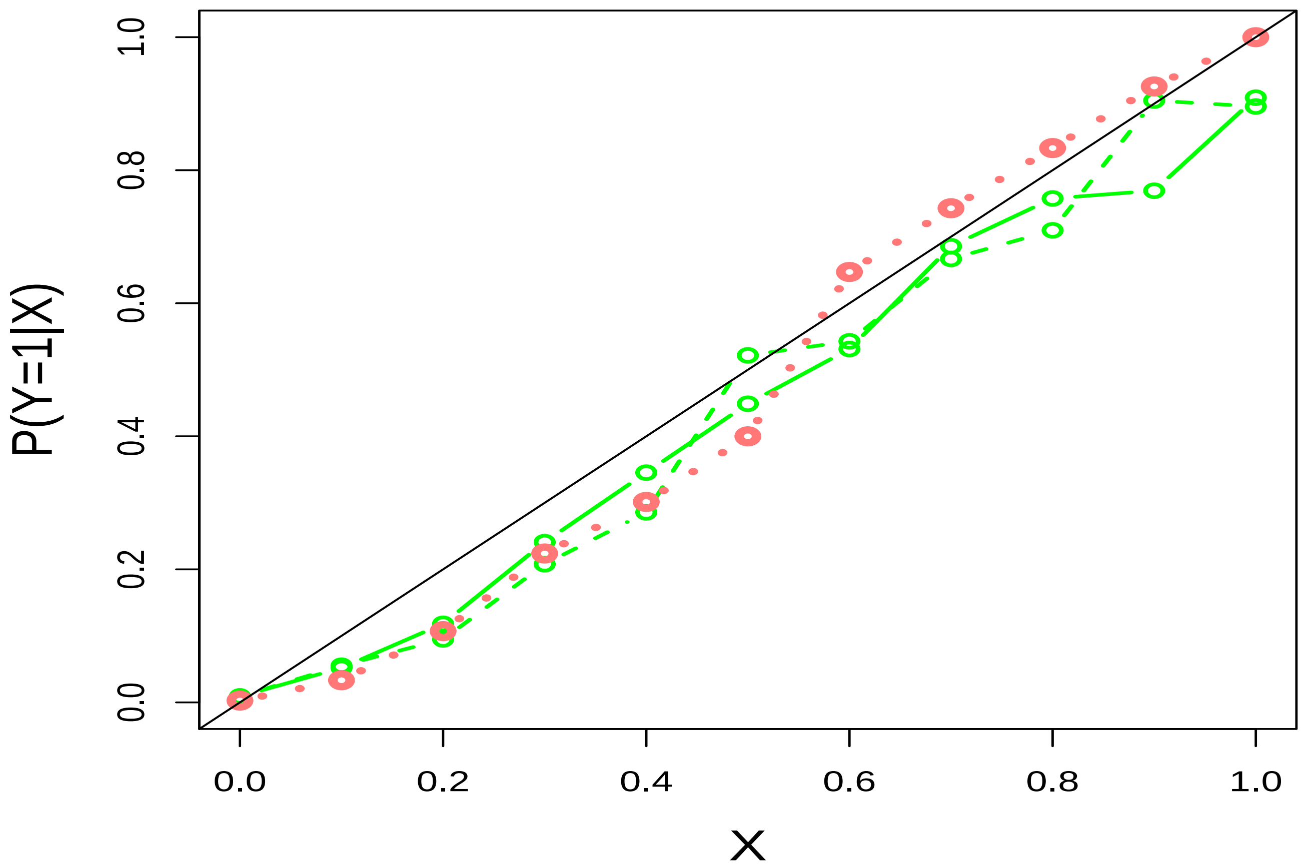
Y=0
Y=1
A−B
B
OK
1.0
0.5
0.0
−0.5
−1.0
0.0
0.4
0.8



A
B
0.35
0.40
0.45
K̂
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
w



0.0
0.1
0.2
0.3
0.4
0.5
0.6
$\hat{K}$
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Best w

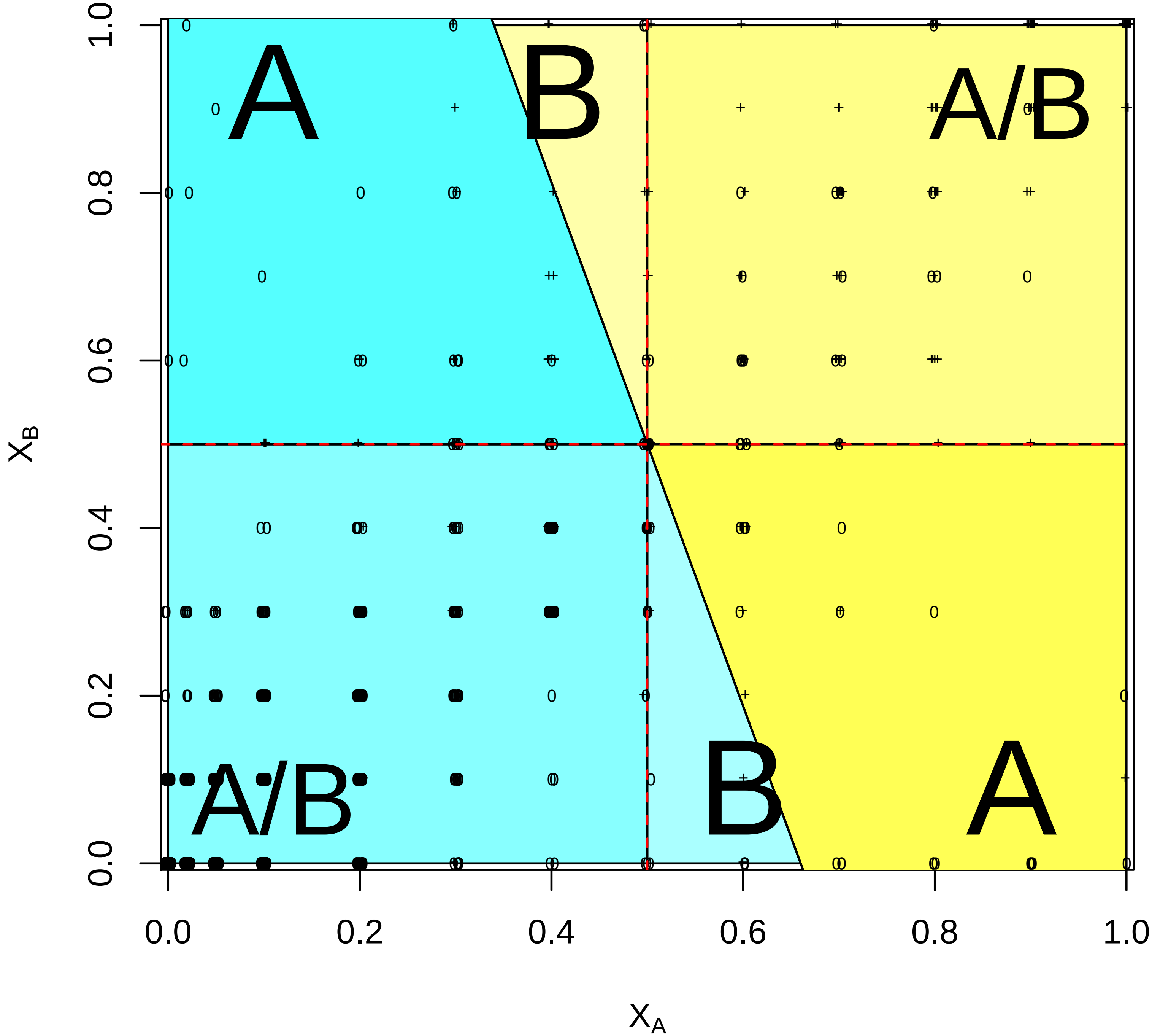


$$\widehat{K}_A = 0.33, \widehat{K}_B = 0.40, \widehat{K}_X = 0.43$$

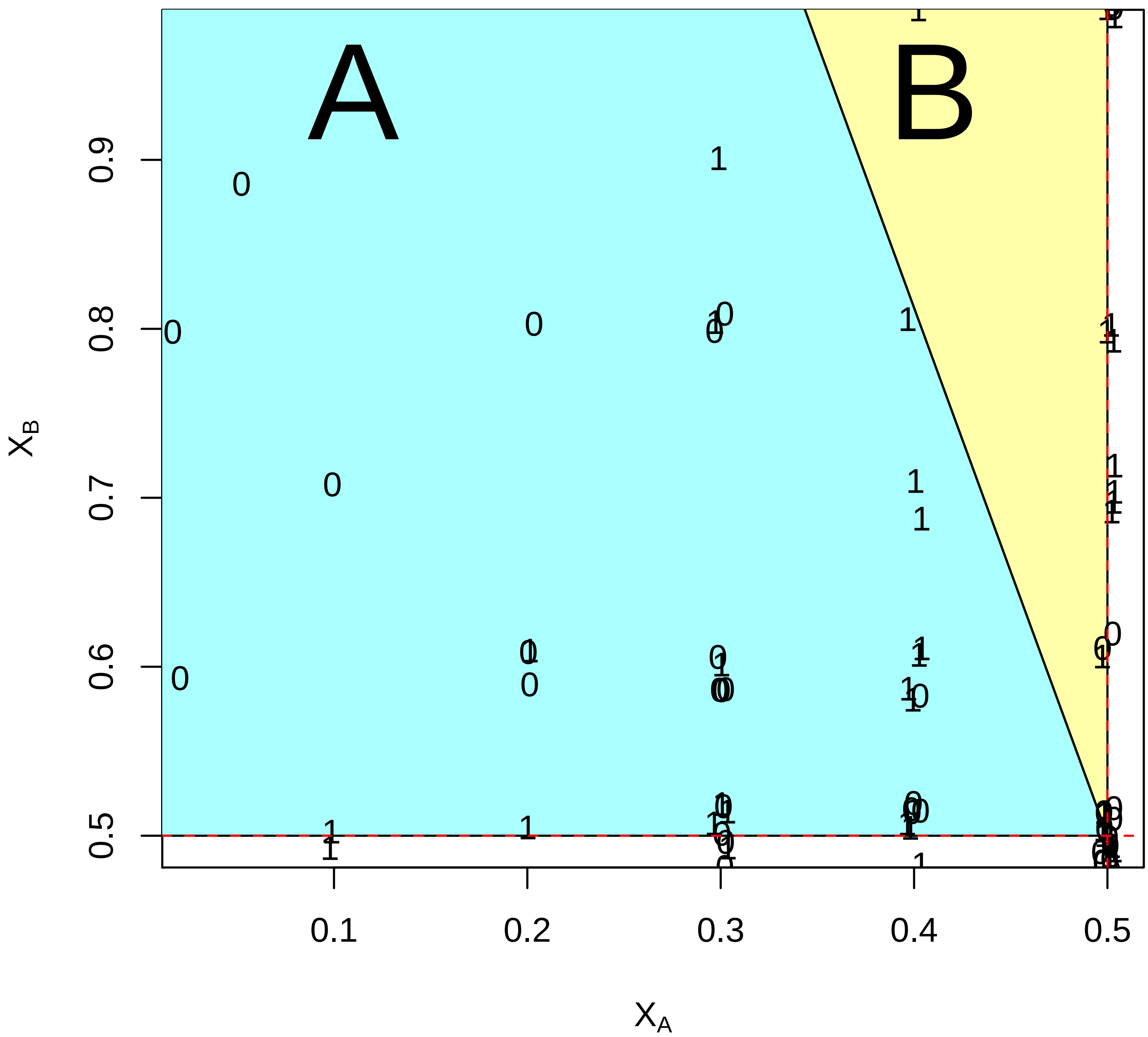
Actual forecast

$$X = I\left(wA + (1-w)B > \gamma\right)$$

$$X = I\left(B > \frac{\gamma}{(1-w)} - \frac{w}{(1-w)}A\right)$$



A
B
A/B
A/B
B
A
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
X_B
X_A


A
B
0.5
0.6
0.7
0.8
0.9
0.1
0.2
0.3
0.4
0.5
X_A
X_B