Robust Parameter Design with Measurement Systems

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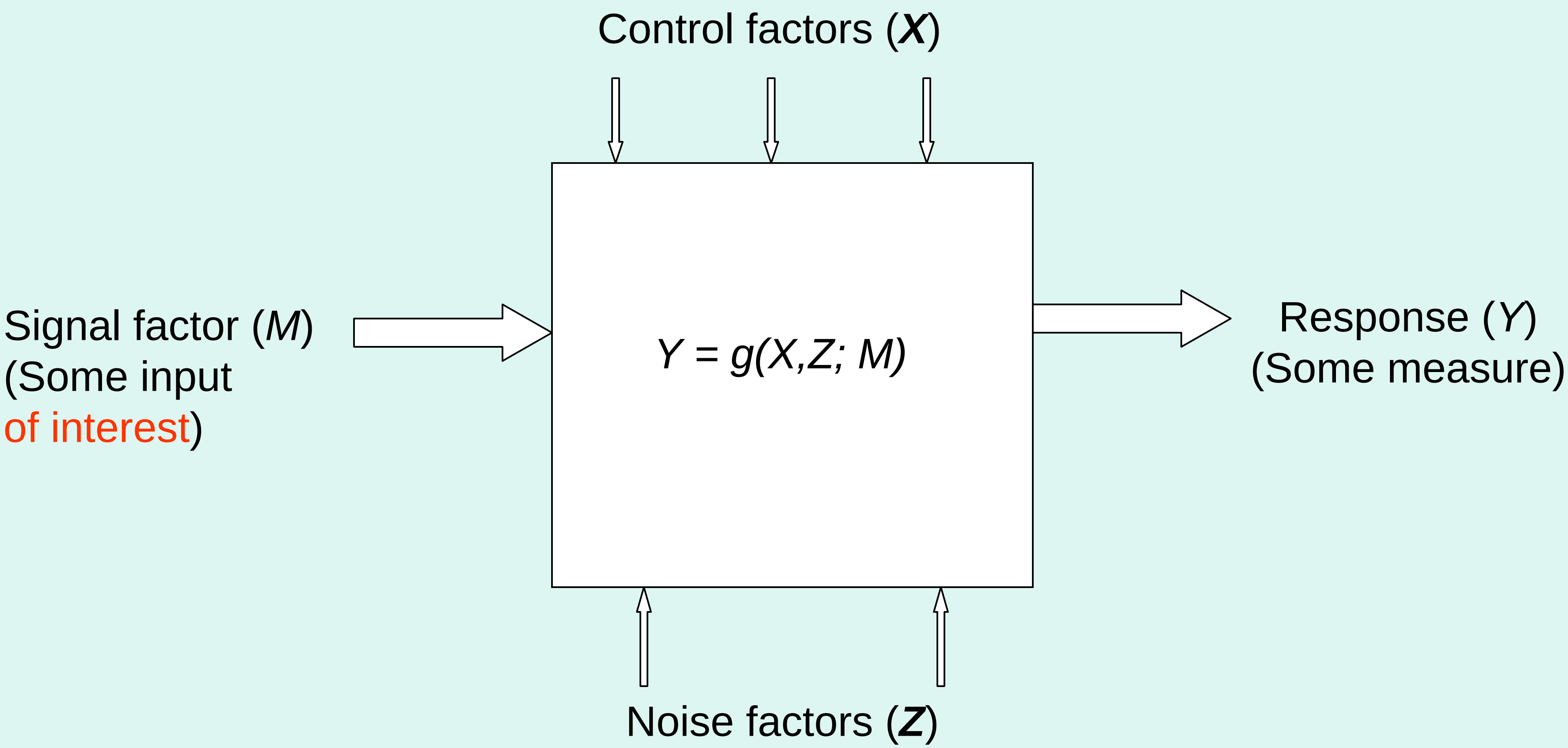
A schematic measurement system



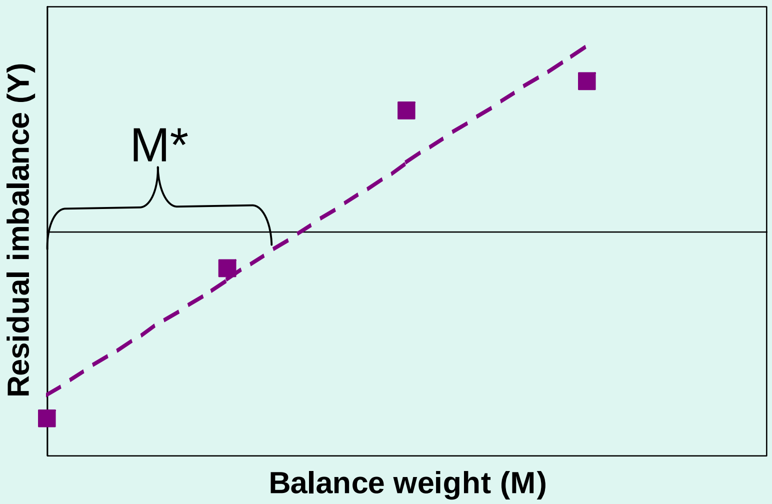
The typical problem (without observable noise *N*)

Control factors	Signal factor (*M*) levels					
	M_1	M_2	M_3	M_4	….	M_k
(X)	Y_{11}	Y_{12}	Y_{13}	Y_{14}	….	Y_{1k}
	Y_{21}	Y_{22}	Y_{23}	Y_{24}	….	Y_{2k}

- $Y = g(X, Z, M) = \alpha(X) + \beta(X)M + \sigma(X)\varepsilon$, where $\varepsilon \sim N(0,1)$.
- $\hat{M} = \left(Y_{obs} - \alpha(X)\right)/\beta(X)$.
- Obtain X such that $var(\hat{M}) = \sigma^2(X)/\beta^2(X)$ is minimized.
- Estimated performance measure (to be maximized) is $\hat{\beta}^2(X)/\hat{\sigma}^2(X)$ (Rigorous proof by Miller and Wu 1996).

Example : Taguchi's drive shaft experiment

- The measurement system : compensates imbalance (*Y*) by attaching balance weights (*M*).
- Purpose of experiment : Achieving system robustness.
- *Y* : Measured residual imbalance.
- *M* : Balance weight attached to drive shaft.
- Noise factor : Variation in imbalance levels of different shafts.
- Control factors : Factors associated with the imbalance tester.



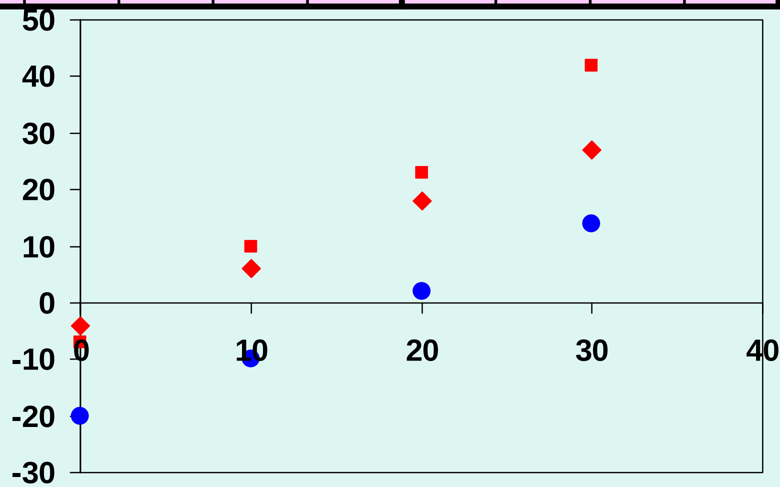
$$\hat{Y} = \hat{\alpha} + \hat{\beta} M,$$

$$\hat{M} = (Y_{obs} - \hat{\alpha}) / \hat{\beta}.$$

M^* : Value of M corresponding to $E(Y) = 0$.

Drive shaft experiment

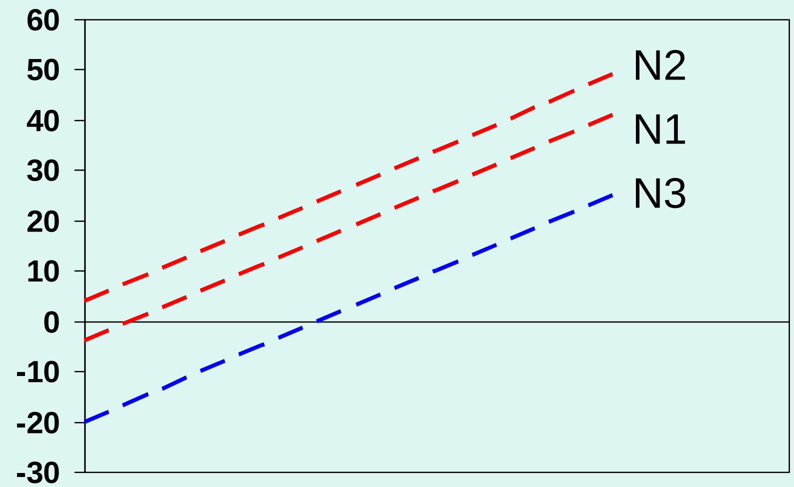
CONTROL ARRAY	N1 : Good Shaft				N2 : Good Shaft				N3 : Bad Shaft			
	0	10	20	30	0	10	20	30	0	10	20	30
	-4	6	18	27	-4	6	15	25	-20	-10	2	14
	…	…	…	…	…	…	…	…	…	…	…	…
(X)	…	…	…	…	…	…	…	…	…	…	…	…



A simple model

CONTROL ARRAY	N1 : Good Shaft				N2 : Good Shaft				N3: Bad Shaft			
	M_1	M_2	M_3	M_4	M_1	M_2	M_3	M_4	M_1	M_2	M_3	M_4
	Y_{11}	Y_{12}	Y_{13}	Y_{14}	Y_{21}	Y_{22}	Y_{23}	Y_{24}	Y_{31}	Y_{32}	Y_{33}	Y_{34}
👎(X)	…	…	…	…	…	…	…	…	…	…	…	…

$$
\begin{aligned}
Y_{jk} &= \alpha_k + \beta M_j + \varepsilon_{jk}, \\
j &= 1,2,3,4, \;\; k = 1,2,3 \\
\varepsilon_{jk} &\sim N(0, \sigma^2).
\end{aligned}
$$



Taguchi's analysis

CONTROL ARRAY	N1 : Good Shaft				N2 : Good Shaft				N3: Bad Shaft				SN ratio
	M_1	M_2	M_3	M_4	M_1	M_2	M_3	M_4	M_1	M_2	M_3	M_4	
(X)	Y_{11}	Y_{12}	Y_{13}	Y_{14}	Y_{21}	Y_{22}	Y_{23}	Y_{24}	Y_{31}	Y_{32}	Y_{33}	Y_{34}	η
	…	…	…	…	…	…	…	…	…	…	…	…	

- Dynamic SN ratio $\eta = \frac{\hat{\beta}^2}{s^2}$,
- $s^2 = \hat{\sigma}^2 = \sum_j \sum_k (Y_{jk} - \hat{\alpha}_k - \hat{\beta} M_j)$,
- $\hat{\alpha}_k$ and $\hat{\beta}$ are least squares estimates.

- Find out which control factors affect η significantly.
- Choose the setting that maximizes η.
- SN Ratio, a special case of performance measure modeling.

Proposed Approach : Summary

- Cross Array Design.

- Two analysis approaches (Miller and Wu, 1996)
  - Response function modeling (RFM)
    - Separate modeling of β and $log\ s^2$
  - Performance measure modeling (PMM)
    - Direct modeling of the performance measure

- Using split-plot techniques in response function modeling.

- Maximum likelihood method for estimation of variance components.

- Optimize system by maximizing performance measure.

Design and model for RFM

Row No.	COMBINED ARRAY (CONTROL & NOISE)	Signal factor level			
			j-th (M_j)		
i-th	$Z =$ 👎(X)⮩ 👎(N)		$\boldsymbol{Y_{ij}}$		

$$
\begin{aligned}
Y_{ij} &= \alpha_i + \beta_i M_j + \sigma_i \varepsilon_{ij}, \\
\beta_i &= \mathbf{z}_i' \Omega_\beta + \sigma_\beta \tau_i. \\
\ln(\sigma_i^2) &= \mathbf{z}_i' \Omega_\sigma + \sigma_\sigma \zeta_i.
\end{aligned}
$$

$\varepsilon_{ij}, \ \tau_i, \ \zeta_i$: independent N(0,1) variables.

Estimated response models

$$
\begin{aligned}
\hat{Y}_{ij} &= \hat{\alpha}_i + \hat{\beta}_i M_j, \\
\hat{\beta}_i &= \beta_i + \varepsilon_i \sqrt{(\sigma_i^2 / S_{mm})} \\
&= \mathbf{z}_i' \Omega_\beta + \xi_i \sqrt{\sigma_\beta^2 + \sigma_i^2 / S_{mm}}. \\
\ln(s_i^2) &= \ln(\sigma_i^2) + \varepsilon_i \sqrt{(2/\nu)} \\
&= \mathbf{z}_i' \Omega_\sigma + \xi_i \sqrt{\sigma_\sigma^2 + 2/\nu}.
\end{aligned}
$$

ξ_i, ε_i : independent N(0,1) variables.

RECOMMENDED APPROACH FOR RFM

- Compute $\hat{\beta}$ and s^2 corresponding to each row.
- Use half-normal plots to detect significant effects for β and $\ln(\sigma^2)$.
- Fit separate models for $\hat{\beta}$ and $\ln(s^2)$ in terms of significant control and noise factors.
- Use maximum likelihood method to estimate σ_β^2 and σ_σ^2.
- An appropriate performance measure is $\hat{\beta}^2/(s^2 + \hat{\sigma}_\beta^2)$.
- If $\sigma_\beta \approx 0$, one can use $\ln(\hat{\beta}^2/s^2)$.
- Determine control factor settings that maximize the performance measure.

Control factors for the drive shaft experiment

FACTORS	Levels			
	1	2	3	4
A : Testing machine	A1 : New	A2 : Old	-	-
B : Master Rotor	B1 : # 1	B2 : # 2	B3 : # 3	B4 : #4
C : Rotations at handling time	C1 : Current	C2 : New	-	-
D : Rotations at measurement	D1 : Current	D2 : New	-	-
E : Signal Sensitivity	E1 : 10	E2 : 20	E3 : 30	E4 : 40
F : Sequence of correction of imbalance	F1 : Current	F2 : Reverse	F3 : New # 1	F4 : New # 2
G : Imbalance correction location	G1 : Current	G2 : New	-	-

Control Array

RUN	A	B	C	D	E	F	G
1	1	1	1	1	1	1	1
2	1	2	1	2	2	2	1
3	1	3	2	1	3	3	1
4	1	4	2	2	4	4	1
5	1	2	1	1	3	4	2
6	1	1	1	2	4	3	2
7	1	4	2	1	1	2	2
8	1	3	2	2	2	1	2
9	2	3	1	1	4	2	1
10	2	4	1	2	3	1	1
11	2	1	2	1	2	4	1
12	2	2	2	2	1	3	1
13	2	4	1	1	2	3	2
14	2	3	1	2	1	4	2
15	2	2	2	1	4	1	2
16	2	1	2	2	3	2	2

Experimental data (Flange side)

Run	DS1 M_1	DS1 M_2	DS1 M_3	DS1 M_4	DS2 M_1	DS2 M_2	DS2 M_3	DS2 M_4	DS3 M_1	DS3 M_2	DS3 M_3	DS3 M_4
1	-4	6	18	27	-4	6	15	25	-20	-10	2	14
2	-7	10	23	42	-3	15	32	46	-34	-20	4	16
3	-4	9	22	34	-7	6	18	30	-26	-15	0	12
4	-2	10	22	36	-4	8	22	34	-30	-18	-7	10
5	-6	6	16	28	-5	6	16	27	-21	-12	3	13
6	-7	13	32	50	-7	14	31	48	-45	-27	-8	13
7	-13	10	30	52	-8	10	30	50	-37	-18	7	26
8	-19	(8)	27	48	-14	7	29	52	-42	-25	-2	22
9	-10	4	16	29	-8	6	16	26	-29	-20	-14	[-14]
10	-14	11	32	51	-18	4	25	46	-44	-26	-11	16
11	-3	2	10	16	-4	2	(17)	13	-13	-8	-5	7
12	-5	5	16	25	-7	3	12	22	-22	-14	-8	12
13	-4	6	18	30	-8	2	16	28	-23	-15	-10	18
14	-6	16	38	62	-16	6	32	55	-44	-25	-6	21
15	-4	2	7	14	-4	2	6	12	-10	-6	-3	7
16	-5	6	16	30	-7	4	16	27	-25	-14	-6	14

Preliminary Analysis : Effect of machine

MACHINE (FACTOR A)	N1 : Good Shaft				N2 : Good Shaft				N3 : Bad Shaft			
	M=0	M=10	M=20	M=30	M=0	M=10	M=20	M=30	M=0	M=10	M=20	M=30
1(NEW)												
1(NEW)												
1(NEW)												
1(NEW)												
1(NEW)												
1(NEW)												
1(NEW)												
1(NEW)												
2 (OLD)												
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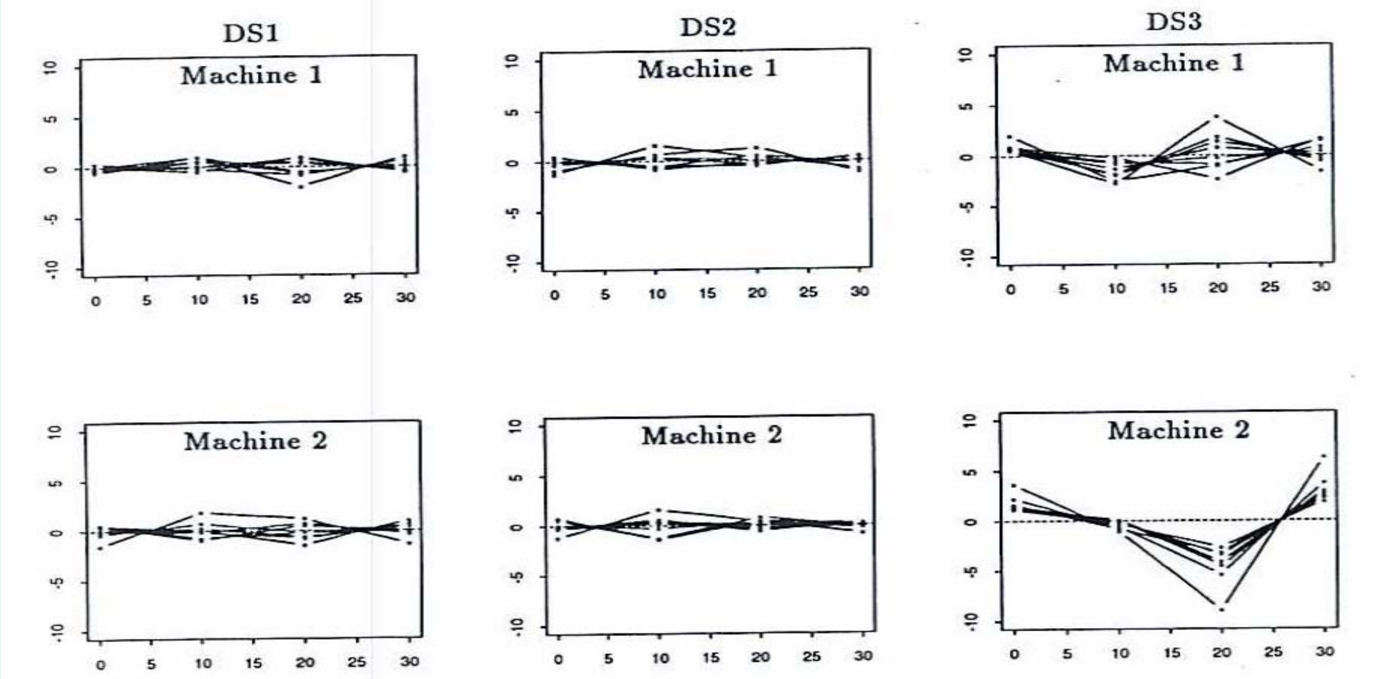
2 (OLD)												

- Fit $Y = \hat{\alpha} + \hat{\beta} M$.
- Compute residuals $e = Y - \hat{\alpha} - \hat{\beta} M$.
- Plot residuals for each machine/shaft combination.

Preliminary analysis (contd.)



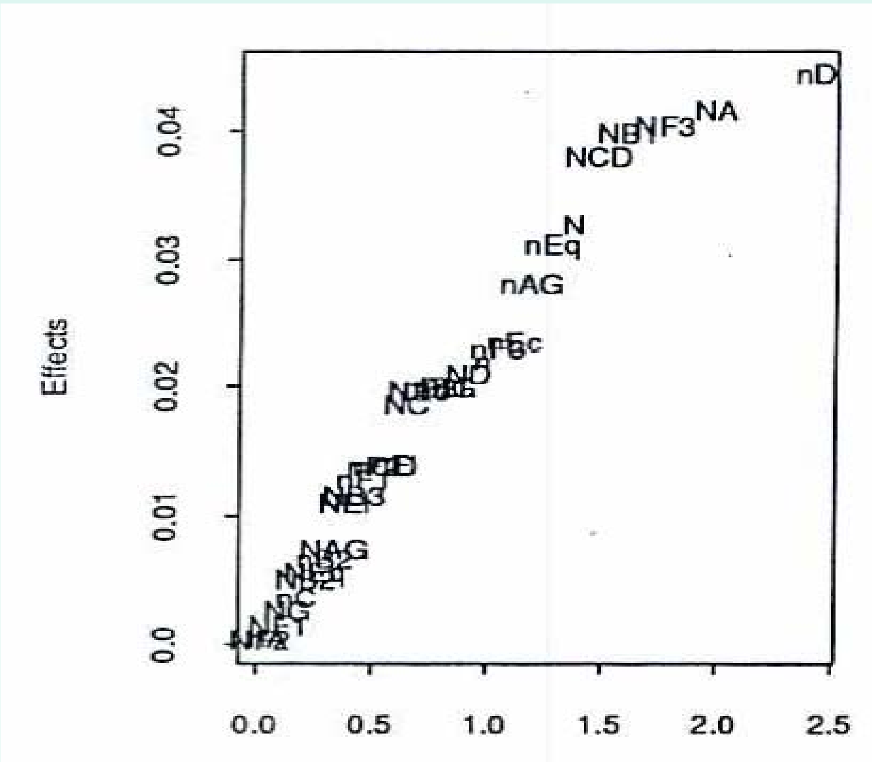
- Large residuals for drive shaft 3.
- Variance component depends on machine x shaft interaction.
- Missed in Taguchi's analysis.
- Such information may be practically useful, (e.g., reducing measurement error by utilizing machine x appraiser interaction.)

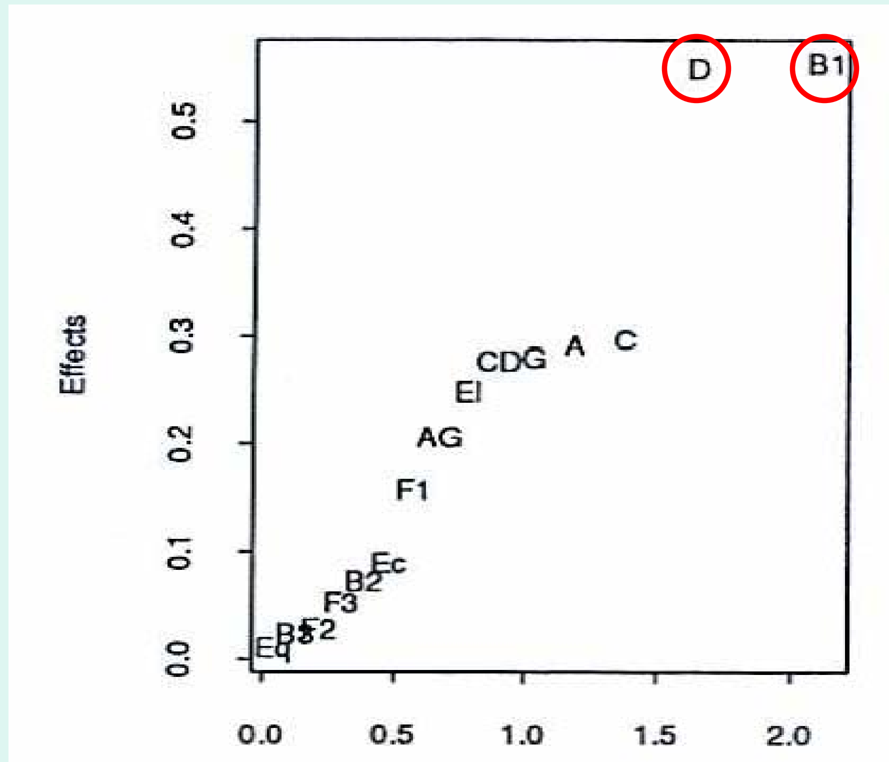
RFM Analysis

RUN	Control factors							Noise (Shaft)	Signal factor				β	s^2
	A	**B**	**C**	**D**	**E**	**F**	**G**		M_1	M_2	M_3	M_4		
1	1	1	1	1	1	1	1	N1						
2	1	1	1	1	1	1	1	N2						
3	1	1	1	1	1	1	1	N3						
…..	…..	…..	…..	…..	…..	…..	…..	…..	…..	…..	…..	…..	…..	…..
48	2	1	2	2	3	2	2	N3						

- Noise factor split into two orthogonal contrasts N and n.
  - n contrasts N1 with N2; N contrasts N3 with N1 and N2.
- Four-level qualitative factors B and F split into three orthogonal contrasts each.
- Four-level quantitative factor E split into three orthogonal contrasts – linear, quadratic, cubic.
- Two groups of estimated effects (split-plot analysis, Box and Jones 1992)
  - Those involving N and n.
  - Those not involving N and n.

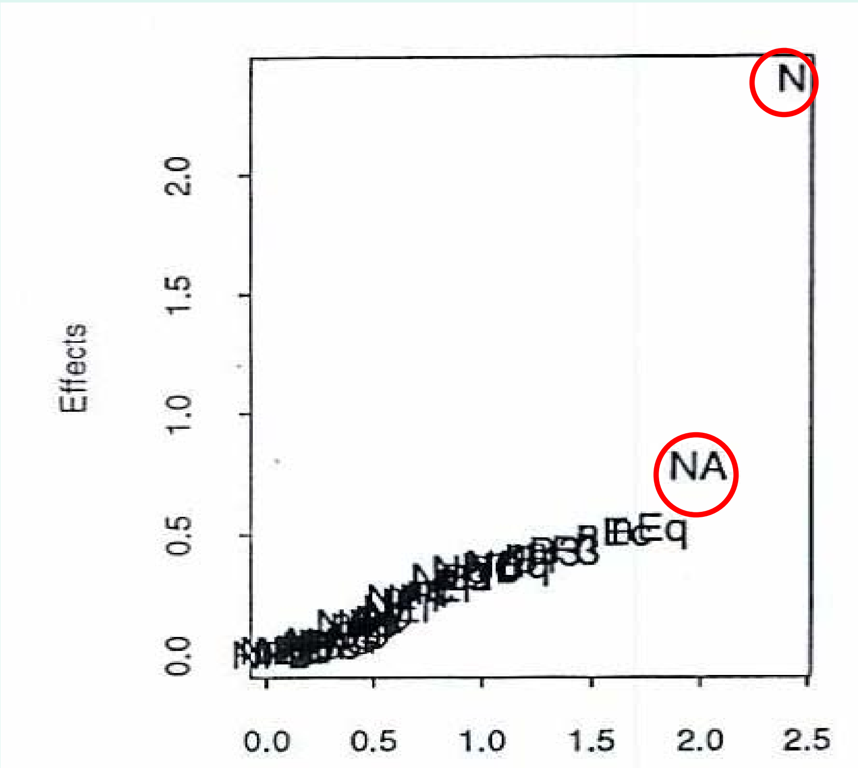
Half-normal plots for β

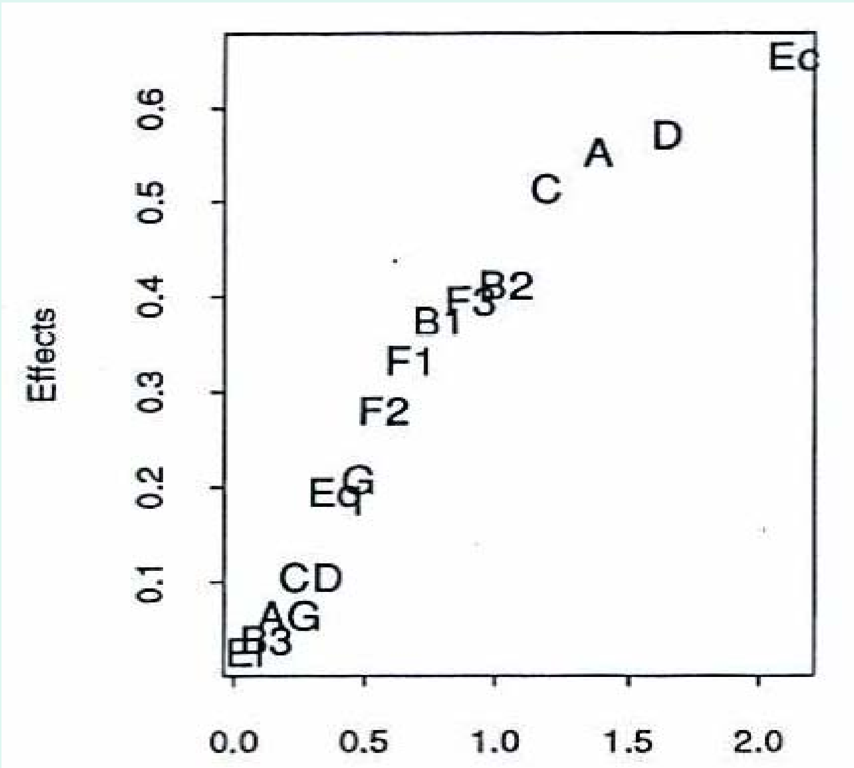




- No effects involving *n,N* significant ($\sigma^2_\beta \approx 0$).
- *D* and B_1 significant.

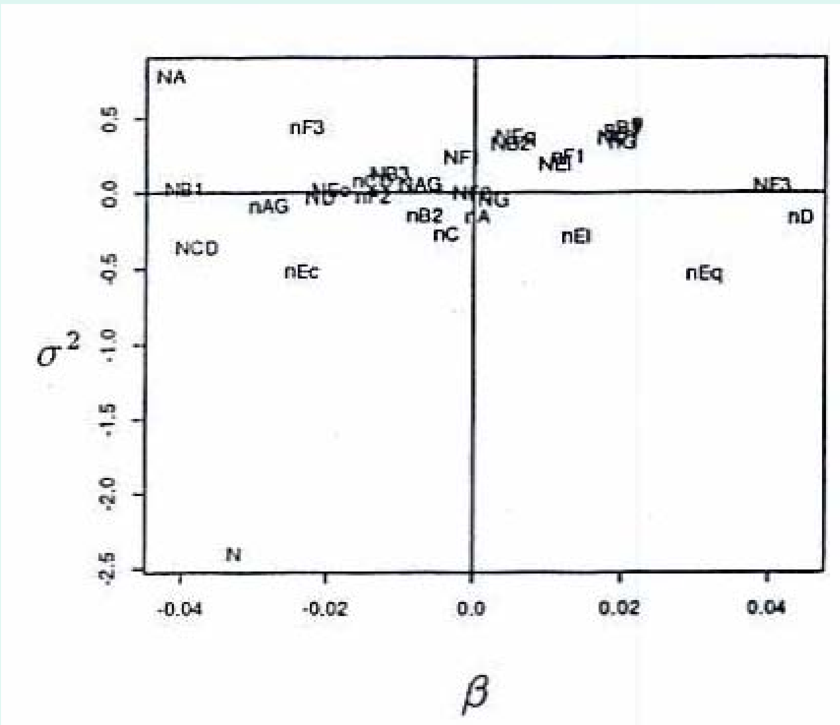
Half-normal plots for ln(σ^2)

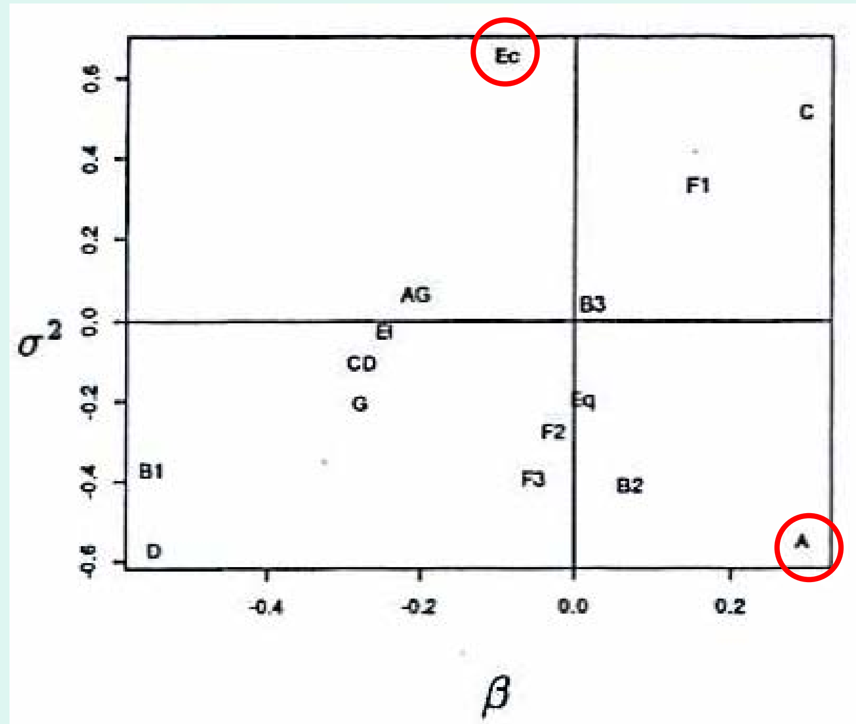




- *N,NA* significant.
- As expected, residual variance is affected by shaft x machine interaction.

Correlation among parameter estimates





- No trend in the first plot; trend in second plot.
- *A* and E_c stand apart - warrant further consideration.
- E_c not considered – difficult to justify inclusion of cubic effect in model without linear and quadratic effects.

Fitted models

$$\hat{\log}\beta^2 = 0.54\ [+0.29X_A] - 0.44X_{B1} - 0.43X_D$$
$$[-0.04X_N - 0.02X_N X_A],$$
$$\hat{\log}\sigma^2 = 0.59\ [-0.42X_A - 0.19X_{B1} - 0.29X_D]$$
$$-1.27X_N + 0.41X_N X_A.$$

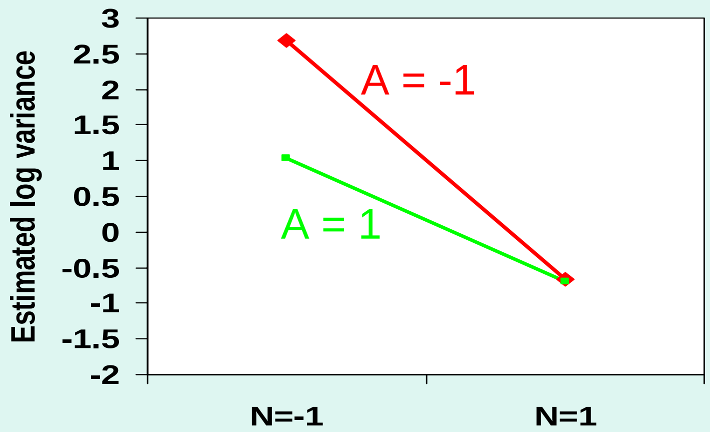
- D, B_1 significant for β.
- N, NA significant for $\ln(\sigma^2)$.
- A is included on the basis of the correlation plot.
- The same set of effects are included in both models (effects in square brackets are those not found significant for that parameter).

Optimal factor settings

- Maximize $\hat{PM} = \hat{\log}\beta^2 - \hat{\log}\sigma^2$
$= -0.05 + 0.71X_A - 0.25x_{B1} - 0.14X_D - 1.23X_N - 0.39X_N X_A.$

- The above is maximized if
$X_A = 1, \; X_{B1} = -1, \; X_D = -1.$

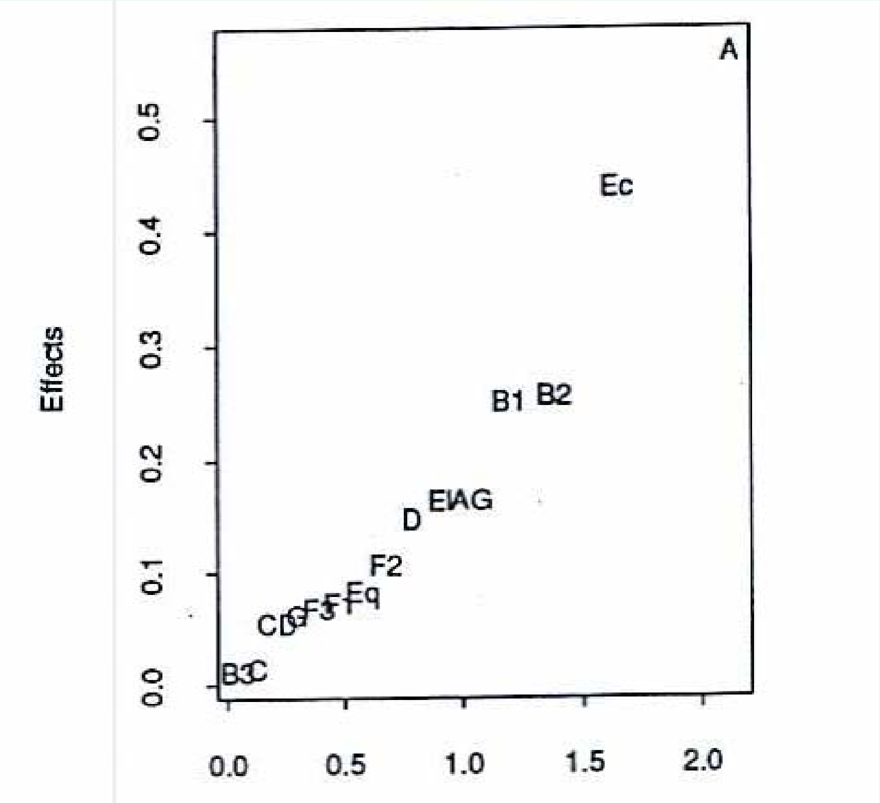
- Thus the optimal combination is $A = A_1$, $B = B_3$ or B_4, $D = D_2$.



- Note : For testing a *good* shaft ($N = 1$) either machine is OK. However, for testing a *bad* shaft ($N = -1$), the new machine ($A = 1$) is much better.

PMM (SN Ratio) with drive shaft data

	DS1		DS2		DS3		
Run	$\hat{\beta}$	s^2	$\hat{\beta}$	s^2	$\hat{\beta}$	s^2	$\hat{PM}_i$
1	1.05	0.75	0.96	0.10	1.14	0.60	1.13
2	1.60	3.00	1.64	2.10	1.74	12.60	-0.45
3	1.27	0.15	1.23	0.15	1.29	1.35	1.63
4	1.26	0.60	1.28	0.40	1.31	4.35	0.48
5	1.12	0.40	1.06	0.10	1.17	3.15	0.91
6	1.90	0.50	1.82	2.40	1.93	1.15	1.16
7	2.15	0.75	1.94	0.60	2.14	3.60	1.30
8	2.00	0.67	2.20	0.50	2.15	6.75	1.23
9	1.29	0.35	1.12	2.40	1.02	12.90	-0.54
10	2.16	4.60	2.13	0.15	1.95	15.75	0.67
11	0.65	0.75	0.56	0.07	0.63	9.15	-0.74
12	1.01	0.35	0.96	0.10	1.08	24.40	0.08
13	1.14	0.60	1.22	1.40	1.28	66.90	-0.96
14	2.26	0.60	2.39	1.35	2.14	9.60	0.95
15	0.59	0.35	0.52	0.40	0.54	6.10	-1.15
16	1.15	1.75	1.14	0.10	1.25	15.75	-0.01



- *A* and *Ec* are marginally significant.

COMPARISON OF RFM AND PMM APPROACHES

SUMMARY OF RFM RESULTS

$$\hat{\log}\beta^2 = 0.54\,[+0.29X_A] - 0.44X_{B1} - 0.43X_D\,[-0.04X_N - 0.02X_N X_A],$$

$$\hat{\log}\sigma^2 = 0.59\,[-0.42X_A - 0.19X_{B1} - 0.29X_D] - 1.27X_N + 0.41X_N X_A.$$

- *A* marginally significant for both models; has opposite signs.
- *D, B1* are significant for the slope; have same signs.
- E_c identified from correlation plots but not considered due to absence of proper engineering justification.
- NA significant for $\ln(s^2)$: leads to practically important conclusions.

SUMMARY OF PMM RESULTS

$$\hat{PM} = \hat{\log}\beta^2 - \hat{\log}\sigma^2 = -\ 0.05 + 0.71X_A - 0.25x_{B1} - 0.14X_D - 1.23X_N - 0.39X_N X_A.$$

PM is directly modelled in this approach.

- Identifies *A* as significant.
- Misses *D, B1.*
- Identifies *Ec* as significant.
- Fails to identify effects involving noise owing to limitations of the analysis strategy.

SUMMARY AND CONCLUSIONS

- A design and analysis strategy for obtaining robust measurement systems is proposed.
- A random coefficients model can be effectively used for modeling and analysis of data.
- Variance components of the model have meaningful practical interpretations.
- The model helps in rigorous justification of each step involved in analysis.
- The RFM approach provides much more insightful results as compared to PMM.

Additional slides

Generalized version with noise factors

Combined Control and Noise Array	Signal factor (*M*) levels					
	M_1	M_2	M_3	M_4	….	M_k
(X,N) [Single array] OR	Y_{11}	Y_{12}	Y_{13}	Y_{14}	….	Y_{1k}
(X) ⇨ (N) [Cross array]	Y_{21}	Y_{22}	Y_{23}	Y_{24}	….	Y_{2k}

- $Y = \alpha(X,N) + \beta(X,N)M + \sigma(X,N)\varepsilon$, where $\varepsilon \sim N(0,1)$.
- $\hat{M} = \left(Y_{obs} - \alpha(X,N)\right)/E_N\left(\beta(X,N)\right)$.
- Obtain X such that $var(\hat{M})$ is minimized.

Questions and Concerns

- Statistical (Model-related) issues
  - Dependence of slope (β) and error variance (σ^2) on noise (shafts) not considered.
  - The model does not consider the fact that the noise (product) is a random factor.
- Engineering issues
  - Dependence of β and σ^2 on control x noise interactions may be of practical interest, for example,
    - A certain group of appraisers may have preference for a certain machine.
    - Incoming products from comparatively unreliable suppliers may be tested in a separate machine.