Optimal Block Sequences for Blocked Fractional Factorial Split-plot Designs

Robert G. McLeod, University of Winnipeg

2007 Quality and Productivity Research Conference, Santa Fe, New Mexico

Outline

1. Basic Concepts of Screening Designs
   - 2^{n-k} fractional factorial (FF) designs
   - blocked 2^{n-k} designs (BFF designs)
   - $2^{(n_1+n_2)-(k_1+k_2)}$ fractional factorial split-plot (FFSP) designs

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   - $2^{(n_1+n_2)-(k_1+k_2)}$ fractional factorial split-plot (FFSP) designs
2. Optimal Block Sequences for Blocked Fractional Factorial Split-plot (BFFSP) Designs
   - approaches to blocking; an example
   - advantages of blocking
   - block sequences
   - optimality criteria
   - a catalog of optimal block sequences for BFFSP designs

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   - constructing non-regular BFFSP designs
   - analysis of BFFSP designs with complex aliasing

1. Basic Concepts of Screening Designs

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A 2^{7-2} Design

Generators & Defining Contrast Subgroup (DCS):

$$F = ABC \qquad G = ABDE$$

$$I = ABCF = ABDEG = CDEFG$$

A 2^{7-2} Design

Generators & Defining Contrast Subgroup (DCS):

$$\mathrm{F} = \mathrm{ABC} \qquad \mathrm{G} = \mathrm{ABDE}$$

$$\mathrm{I} = \mathrm{ABCF} = \mathrm{ABDEG} = \mathrm{CDEFG}$$

Word Length Pattern:

$$\text{WLP} = (W_3, W_4, W_5, \ldots)$$

$$\text{Here, WLP} = (0,1,2)$$

This design is the minimum aberration (MA) 2^{7-2} design (Fries & Hunter, 1980).

Clear Effects

Definition:

> A main effect or two-factor interaction is **clear** if it is not aliased with any main effects or two-factor interactions (or confounded with blocks).

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DCS for the MA 2^{7-2} design:

$$\mathrm{I} = \mathrm{ABCF} = \mathrm{ABDEG} = \mathrm{CDEFG}$$

Clear Effects:

- for the 2^{7-2} design, all main effects and 15 two-factor interactions are clear
- in this example, the MA design also maximizes the number of clear effects; however, this is not always true

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A 2^{5-1} Design in 2 Blocks

Factor Generator:

$$E = ABC$$

Blocking Generator:

$$\beta = ABD$$

Defining Contrast Subgroup:

$$I = ABCE = ABD\beta = CDE\beta$$

A 2^{5-1} Design in 2 Blocks

Run	A	B	C	D	E	Block
1	−	−	−	−	−	1
2	+	+	−	−	−	1
3	−	−	+	−	+	1
4	+	+	+	−	+	1
5	+	−	−	+	+	1
6	−	+	−	+	+	1
7	+	−	+	+	−	1
8	−	+	+	+	−	1
9	+	−	−	−	+	2
10	−	+	−	−	+	2
11	+	−	+	−	−	2
12	−	+	+	−	−	2
13	−	−	−	+	−	2
14	+	+	−	+	−	2
15	−	−	+	+	+	2
16	+	+	+	+	+	2

Table 1: Standard Run Order of a 2^{5-1} Design.

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A $2^{(3+3)-(0+1)}$ FFSP Design

Hard-to-vary (whole-plot) factors; easy-to-vary (subplot) factors

A $2^{(3+3)-(0+1)}$ FFSP Design

Hard-to-vary (whole-plot) factors; easy-to-vary (subplot) factors

Suppose that an experimenter wishes to conduct a split-plot design having $n_1 = 3$ whole-plot (WP) factors and $n_2 = 3$ subplot (SP) factors but can only afford only 32 runs.

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A $2^{(3+3)-(0+1)}$ FFSP Design

Hard-to-vary (whole-plot) factors; easy-to-vary (subplot) factors

Suppose that an experimenter wishes to conduct a split-plot design having $n_1 = 3$ whole-plot (WP) factors and $n_2 = 3$ subplot (SP) factors but can only afford only 32 runs.

The "best" 32-run FFSP design is obtained using the generator $r = ABCpq$.

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A $2^{(3+3)-(0+1)}$ FFSP Design

Run	A	B	C	p	q	r
1	−	−	−	−	−	−
2				+	−	+
3				−	+	+
4				+	+	−
5	+	−	−	−	−	+
6				+	−	−
7				−	+	−
8				+	+	+
9	−	+	−	−	−	+
10				+	−	−
11				−	+	−
12				+	+	+
13	+	+	−	−	−	−
14				+	−	+
15				−	+	+
16				+	+	−
⋮					⋮	

Table 2: Standard run order of the $2^{(3+3)-(0+1)}$ FFSP design—first 16 runs.

2. Optimal Block Sequences for BFFSP Designs

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A $2^{(3+3)-(0+1)\pm(1+1)}$ BFFSP Design

Factor Generator:

$$r = ABq$$

Blocking Generators:

$$\beta = ABC \qquad \text{“a pure WP-blocking generator”}$$

$$\delta = ACpq \qquad \text{“a separator”}$$

Defining Contrast Subgroup:

$$I = ABqr = ABC\beta = ACpq\delta$$

$$= Cqr\beta = BCpr\delta = Bpq\beta\delta = Apr\beta\delta$$

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Defining Contrast Subgroup:

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$$= Cqr\beta = BCpr\delta = Bpq\beta\delta = Apr\beta\delta$$

Clear Effects:

- all main effects and 9 two-factor interactions are clear
- 6 two-factor interactions are not clear: AB, Aq, Ar, Bq, Br & qr

This design is the MA BFFSP design, among all designs having 3 WP factors, 3 SP factors and a (4:4:2) “structure” (i.e., 4 blocks, 4 WP’s per block and 2 SP’s per WP); see McLeod & Brewster (2004).

A $2^{(3+3)-(0+1)\pm(1+1)}$ BFFSP Design

Run	A	B	C	p	q	r	β	δ	Block
1	+	−	−	+	−	+	−	−	1
2				−	+	−	−	−	1
3	−	+	−	−	−	+	−	−	1
4				+	+	−	−	−	1
5	+	−	+	−	−	+	−	−	1
6				+	+	−	−	−	1
7	−	+	+	+	−	+	−	−	1
8				−	+	−	−	−	1
9	−	−	−	+	−	−	+	−	2
10				−	+	+	+	−	2
11	+	+	−	−	−	−	+	−	2
12				+	+	+	+	−	2
13	−	−	+	−	−	−	+	−	2
14				+	+	+	+	−	2
15	+	+	+	+	−	−	+	−	2
16				−	+	+	+	−	2
⋮						⋮			⋮

Table 3: Standard run order of the $2^{(3+3)-(0+1)\pm(1+1)}$ BFFSP Design—first 16 runs.

Advantages of Blocking

An often overlooked advantage of blocked designs, in general, lies in the sequential nature in which they are run.

Basic Concepts

Advantages of Blocking

An often overlooked advantage of blocked designs, in general, lies in the sequential nature in which they are run.

Utilizing the sequential nature of the design. . .

- interim analysis
- early termination of the experiment
- more sophisticated design and analysis techniques

[Daniel (1962); Bisgaard (1994); McLeod and Brewster (2004); Jacroux (2006)]

Block Sequences

Recall the $2^{(3+3)-(0+1)\pm(1+1)}$ BFFSP design with blocking variable generators $\beta = ABC$ and $\delta = ACpq$.

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Let $ABC =$ "–" and $ACpq =$ "–" denote those runs in the BFFSP design which produce a minus sign in the contrasts generating β and δ, respectively.

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Block Sequences

Recall the $2^{(3+3)-(0+1)\pm(1+1)}$ BFFSP design with blocking variable generators $\beta = ABC$ and $\delta = ACpq$.

Let $ABC =$ "$-$" and $ACpq =$ "$-$" denote those runs in the BFFSP design which produce a minus sign in the contrasts generating β and δ, respectively.

Similarly, let $ABC =$ "$+$" and $ACpq =$ "$+$" denote those runs in the BFFSP design which produce a plus sign in the contrasts generating β and δ, respectively.

Block Sequences

	$ABC\beta$ $(Cqr\beta)$	$ACpq\delta$ $(BCpr\delta)$	$Bpq\beta\delta$ $(Apr\beta\delta)$
S_1:			
Block 1	$-$	$-$	$+$
Block 2	$-$	$+$	$-$
Block 3	$+$	$+$	$+$
Block 4	$+$	$-$	$-$

Table 4: One possible block sequence, S_1, for the MA $2^{(3+3)-(0+1)\pm(1+1)}$ BFFSP design.

Block Sequences

	$ABC\beta$ $(Cqr\beta)$	$ACpq\delta$ $(BCpr\delta)$	$Bpq\beta\delta$ $(Apr\beta\delta)$
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Block 4	$+$	$-$	$-$

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$$ABqr \in G_t$$

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$ABqr \in G_t$

$ABC\beta$, $ACpq\delta$ and $Bpq\beta\delta \in G_b$

Block Sequences

	$ABC\beta$ ($Cqr\beta$)	$ACpq\delta$ ($BCpr\delta$)	$Bpq\beta\delta$ ($Apr\beta\delta$)
S_1:			
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$ABqr \in G_t$

$ABC\beta$, $ACpq\delta$ and $Bpq\beta\delta \in G_b$

$Cqr\beta$, $BCpr\delta$ and $Apr\beta\delta \in G_{b\times t}$

Block Sequences

For a given BFFSP design there are $(2^{b_1+b_2})!$ possible block sequences.

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For the $2^{(3+3)-(0+1)\pm(1+1)}$ BFFSP design this implies there are $(2^{1+1})! = 24$ block sequences.

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Block Sequences

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For the $2^{(3+3)-(0+1)\pm(1+1)}$ BFFSP design this implies there are $(2^{1+1})! = 24$ block sequences.

	$ABC\beta$ $(Cqr\beta)$	$ACpq\delta$ $(BCpr\delta)$	$Bpq\beta\delta$ $(Apr\beta\delta)$
S_1:			
Block 1	$-$	$-$	$+$
Block 2	$-$	$+$	$-$
Block 3	$+$	$+$	$+$
Block 4	$+$	$-$	$-$
S_2:			
Block 1	$-$	$-$	$+$
Block 2	$+$	$-$	$-$
Block 3	$-$	$+$	$-$
Block 4	$+$	$+$	$+$

Table 5: A comparison of *two* block sequences for the MA $2^{(3+3)-(0+1)\pm(1+1)}$ BFFSP design.

Block Sequences

Question:

Why concern oneself with block sequences?

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Short answer:

From an estimation perspective, not all block sequences are “created equal”!

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Short answer:

From an estimation perspective, not all block sequences are “created equal”!

Longer answer:

The choice of block sequence may allow, or conversely impede, early estimation of low-order effects in $\mathrm{G}_{\mathrm{b}\times\mathrm{t}} \cup \mathrm{G}_{\mathrm{b}}$.

This realization is critical if one is interested in interim data analysis. . .

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Block Sequences: A comparison of S_1 and S_2

The two block sequences, S_1 and S_2, are *not* equivalent with respect to the early estimation of low-order effects in $\mathrm{G}_{\mathrm{b}\times\mathrm{t}} \cup \mathrm{G}_{\mathrm{b}}$.

Block Sequences: A comparison of S_1 and S_2

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To see this, consider the sign of the contrasts ABC and Cqr (both confounded with β) in S_1 and S_2.

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To see this, consider the sign of the contrasts ABC and Cqr (both confounded with β) in S_1 and S_2.

In S_1 the sign of ABC and Cqr remains constant ("–") thru blocks 1 and 2.

Block Sequences: A comparison of S_1 and S_2

The two block sequences, S_1 and S_2, are *not* equivalent with respect to the early estimation of low-order effects in $G_{b \times t} \cup G_b$.

To see this, consider the sign of the contrasts ABC and Cqr (both confounded with β) in S_1 and S_2.

In S_1 the sign of ABC and Cqr remains constant ("–") thru blocks 1 and 2.

Conversely, in S_2 the sign of ABC and Cqr switches between blocks 1 and 2.

Block Sequences: A comparison of S_1 and S_2

Consequence:

After block 2, using S_1: Two SP main effects (q and r) and all three WP main effects are (completely) aliased with 2fi's after the second block has been completed.

Block Sequences: A comparison of S_1 and S_2

Consequence:

After block 2, using S_1: Two SP main effects (q and r) and all three WP main effects are (completely) aliased with 2fi's after the second block has been completed.

That is, after block 2, $q = Cr$, $r = Cq$, $A = BC$, $B = AC$ and $C = AB = qr$.

Block Sequences: A comparison of S_1 and S_2

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That is, after block 2, $q = Cr$, $r = Cq$, $A = BC$, $B = AC$ and $C = AB = qr$.

After block 2, using S_2: All main effects in $G_{b \times t} \cup G_b$ are clear. Furthermore, no 2fi's in $G_{b \times t} \cup G_b$ are aliased with main effects.

Block Sequences: A comparison of S_1 and S_2

Consequence:

After block 2, using S_1: Two SP main effects (q and r) and all three WP main effects are (completely) aliased with 2fi's after the second block has been completed.

That is, after block 2, $q = Cr$, $r = Cq$, $A = BC$, $B = AC$ and $C = AB = qr$.

After block 2, using S_2: All main effects in $G_{b \times t} \cup G_b$ are clear. Furthermore, no 2fi's in $G_{b \times t} \cup G_b$ are aliased with main effects.

Using the hierarchical principle, S_2 is preferred.

Block Sequences: A comparison of S_1 and S_2

After block 4, using any S_i:

Note that after the runs in the final block (block 4) have been conducted, no low-order treatment effects in $G_{b \times t} \cup G_b$ will be aliased with one another.

Optimality Criteria

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For a given BFFSP design, the idea is select a block sequence that allows the low-order effects in $G_{b\times t} \cup G_b$ to be clearly estimable **as soon as possible**.

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All $(2^{b_1+b_2})!$ possible block sequences are evaluated according to the following sequential ranking scheme:

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For a given BFFSP design, the idea is select a block sequence that allows the low-order effects in $G_{b\times t} \cup G_b$ to be clearly estimable **as soon as possible**.

All $(2^{b_1+b_2})!$ possible block sequences are evaluated according to the following sequential ranking scheme:

a: the # of SP me's that are aliased with other me's

b: the # of WP me's that are aliased with other me's

c: the # of SP me's that are aliased with 2fi's

d: the # of WP me's that are aliased with 2fi's

e: the # of 2fi's involving at least one SP factor (WP×SP and SP×SP) that are aliased with me's

f: the # of WP×WP 2fi's that are aliased with me's

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A Catalog of Optimal Block Sequences

Example: An MA $2^{(2+5)-(0+2)\pm(1+2)}$ BFFSP design

A Catalog of Optimal Block Sequences

Example: An MA $2^{(2+5)-(0+2)\pm(1+2)}$ BFFSP design

An optimal block sequence for a 32-run BFFSP design run in 8 blocks of size 4 is as follows:

Design	Optimal Block Sequence	#	a	b	c	d	e	f
2,5;0,2;1,2	$AB\beta_1(-,-,+,+,-,-,+,+)$ $Apr\delta_1(-,-,-,-,+,+,+,+)$ $pq\delta_2(-,+,-,+,-,+,-,+)$	768	0,0,0,0	2,0,0,0	5,4,4,0	2,1,1,0	16,16,6,0	0,0,0,0

3. Discussion and Future Research

Discussion and Future Research

- Analysis Issues

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- Analysis Issues
  - partial ("complex") aliasing
  - simplification of complex aliasing via effect sparsity (Wu and Hamada (2000))
  - Bayesian variable selection strategy (Hamada and Wu (1992); Chipman, Hamada, Wu (1997))
  - in the split-plot setting? Ongoing work by D. Bingham

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Discussion and Future Research

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- Constructing Non-regular Blocked Split-plot Designs
  - an indirect approach is possible using the optimal block sequence catalog
  - 12 and 24-run designs

Optimal Block Sequences for Blocked Fractional Factorial Split-plot Designs

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2007 Quality and Productivity Research Conference, Santa Fe, New Mexico

Block Sequences: A general result

If a 3fi, say $W_1W_2W_3$, is confounded with blocks (for e.g., $W_1W_2W_3 = \beta$, etc.) then the main effects, W_1, W_2 and W_3 will be (completely or partially) aliased with the three 2fi's, W_2W_3, W_1W_3 and W_1W_2, respectively, after a given block, if an unequal number of "–" and "+" signs exist in the $W_1W_2W_3$ contrast.

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If a 3fi, say $W_1W_2W_3$, is confounded with blocks (for e.g., $W_1W_2W_3 = \beta$, etc.) then the main effects, W_1, W_2 and W_3 will be (completely or partially) aliased with the three 2fi's, W_2W_3, W_1W_3 and W_1W_2, respectively, after a given block, if an unequal number of "–" and "+" signs exist in the $W_1W_2W_3$ contrast.

Similar results for contrasts of the form W_1W_2 and $W_1W_2W_3W_4$ confounded with blocks (i.e., in $G_{b\times t} \cup G_b$).