Tolerance Intervals
Time to Revisit

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What is a Tolerance Interval?

- Example of a tolerance interval statement:
  - "I am 90% confident that at least .95 of my data are in the interval (93, 106)"

Coverage, proportion

Another way to look at Confidence...

- Confidence is the percent of times the computed interval has the property you think it has.

Tolerance intervals are of the form $(m - K*s, m + K*s)$

- Assuming a normal distribution
- K is based on 3 parameters
  - P (proportion, coverage)
  - λ (Confidence)
  - n (sample size)
- m is the sample mean, estimating μ
- s is the sample standard deviation, estimating σ

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Looking for multiplier ‘K’

- Experiment: Let K=t
    - (why? Because m+/-t*s/sqrt(n) is the conf. interval for the mean, n=1?)
  - Presume to cover .95 of the data:
  - m +/- t*s
  - n=10
- *I would expect this interval to contain more than .95 about “half the time”, and less than .95 about “half the time”*
- Simulate
  - 10000 samples of size n =10
  - Distribution: standard normal (μ=0, σ=1)
  - t =student’s t, 9 df, .95 two-sided = 2.262

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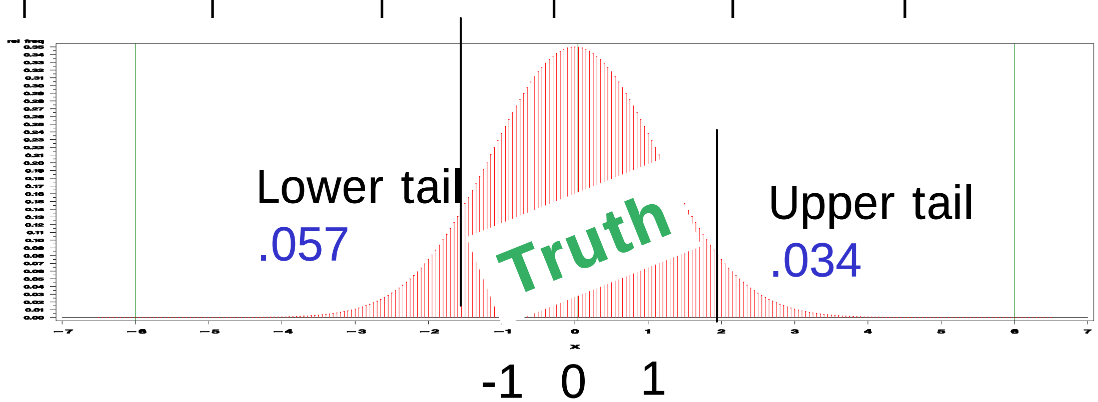
As we run the simulations, we compute data like this.

A success is when the ‘true’ interval coverage is $\geq$.95

run	Mean of 10	std	t	m +/- t*s Lower limit	Upper limit	Lower area	Upper area	Total Tail area	Interval Cover-Age
1	0.23	0.81	2.262	-1.582	1.820	.057	0.034	0.091	0.909*
2	0.50	0.90	2.262	-1.536	2.536	.062	0.006	0.068	0.932*
3	0.19	1.13	2.262	-2.366	2.746	.009	.0030	.0120	0.988
4	etc								
.									
10000									

No too small

yes



Simulation – Results of 10,000 runs

‘t’ contains	out of 10000
less than .95	4056
.95 or more	5944

Success rate= confidence

59.44% contain at least .95 of the data

In tolerance language:

“*For n=10, ‘t’ yields an interval containing at least*
.95 (PROPORTION) of the data
59.4% (CONFIDENCE) of the time”

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Well!! I expected 50%.

- Another idea – use z=1.96
  - – too small, confidence = 37%
- “Right Answer”? Use K=2.135 (not 2.262 or 1.96)
  - m +/- 2.135*s
  - Re-doing simulation shows ~50% of these intervals contain at least .95 of the distribution
- This would be a 50% tolerance interval to cover .95 of the data
- So…where did this K come from?

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Two resources for ‘K’ multipliers of a tolerance interval

- 1. Book/Tables
  - CRC Handbook of Tables for Probability and Statistics, 2nd Ed., CRC Press, Inc., Boca Raton FL, 1987 printing of 1968 edition (Cites Natrella);
- 2. NIST Website for Tolerance Intervals
  - http://www.itl.nist.gov/div898/handbook/prc/section2/prc263.htm

Mary Gibbons Natrella (1963). ***Experimental Statistics,*** NBS Handbook 91, US Deptartment of Commerce.

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Snapshot of CRC Table for K: available λ =75, .90, .95, .99

TOLERANCE FACTORS FOR NORMAL DISTRIBUTIONS

$\lambda = 0.75$

N \ P	0.75	0.90	0.95	0.99	0.999	N \ P	0.75	0.90	0.95	0.99	0.999
2	4.498	6.301	7.414	9.531	11.920	55	1.249	1.785	2.127	2.795	3.571
3	2.501	3.538	4.187	5.431	6.844	60	1.243	1.778	2.118	2.784	3.556
4	2.035	2.892	3.431	4.471	5.657	65	1.239	1.771	2.110	2.773	3.543
5	1.825	2.599	3.088	4.033	5.117	70	1.235	1.765	2.104	2.764	3.531
6	1.704	2.429	2.889	3.779	4.802	75	1.231	1.760	2.098	2.757	3.521
7	1.624	2.318	2.757	3.611	4.593	80	1.228	1.756	2.092	2.749	3.512
8	1.568	2.238	2.663	3.491	4.444	85	1.225	1.752	2.087	2.743	3.504
9	1.525	2.178	2.593	3.400	4.330	90	1.223	1.748	2.083	2.737	3.497
10	1.492	2.131	2.537	3.328	4.241	95	1.220	1.745	2.079	2.732	3.490
11	1.465	2.093	2.493	3.271	4.169	100	1.218	1.742	2.075	2.727	3.484
12	1.443	2.062	2.456	3.223	4.110	110	1.214	1.736	2.069	2.719	3.473
13	1.425	2.036	2.424	3.183	4.059	120	1.211	1.732	2.063	2.712	3.464
14	1.409	2.013	2.398	3.148	4.016	130	1.208	1.728	2.059	2.705	3.456
35	1.283	1.834	2.185	2.871	3.667	900	1.170	1.673	1.993	2.620	3.347
40	1.271	1.818	2.166	2.846	3.635	1000	1.169	1.671	1.992	2.617	3.344
45	1.262	1.805	2.150	2.826	3.609	∞	1.150	1.645	1.960	2.576	3.291
50	1.255	1.794	2.138	2.809	3.588						

NIST formula: ‘k_2’ is the K multiplier for a 2-sided tolerance interval

- You can choose any λ, (they call it ‘ν’ – ”confidence”) or P (proportion, coverage), for any sample size (n) that you want.

$$k_2 = \sqrt{\frac{(N-1)\left(1+\frac{1}{N}\right)z^2_{(1-p)/2}}{\chi^2_{\gamma,N-1}}}$$

Howe, W. G. (1969). *"Two-sided Tolerance Limits for Normal Populations - Some Improvements"*, Journal of the Americal Statistical Association, 64 , pages 610-620.

Magic number?
Parse out algebraically

Upper γ% 1-tailed confidence interval for the std. dev. (Multiplier > 1)

Z*Sqrt(1+1/n) =
Recall the distribution of individual
+ the distribution of the mean,
(for a prediction interval)

$$\mathbf{m} \; \mathbf{+/-} \; s^{*} \sqrt{\frac{(N-1)\boxed{\left(1+\frac{1}{N}\right) z^{2}_{(1-p)/2}}}{\chi^{2}_{\gamma, N-1}}}$$

proportion

Confidence – lower tail*

NIST Example: Compute k_2

$$k_2 = \sqrt{\frac{(N-1)\left(1+\frac{1}{N}\right)z^2_{(1-p)/2}}{\chi^2_{\gamma,N-1}}}$$

- λ=50%, P=.95, n=10
  - $z_{.025}$=1.96,
  - $\chi^2_{(.50,9)} = 8.34$
  - k_2 = sqrt[(10-1)*(1+ 1/10)*1.96^2 /8.34)] = **2.135**

- We are usually interested in higher confidence than 50%. This changes the χ^2 part only
  - For 75% confidence, $\chi^2_{(.75,9)} = 5.90$, $k_2 = 2.39$
  - For 95% confidence, $\chi^2_{(.95,9)} = 3.33$, $k_2 = 3.38$
- If we are interested in better coverage, only the z part changes (k_2 gets larger)
- Larger sample size decreases k_2

Checking Confidence of Tolerance Intervals via Simulations

How good is the k_2 formula? Expanded Simulation Plan

- Use random samples from a normal population with $\mu=0$ and $\sigma=1$
- Compute intervals based on k_2 using
  - Sample size n = 5, 10, 30, 100
  - λ =.75, .90., 95
  - P=.90, .95, .99
- Do 10,000 trials for each combination
- "success" is when an interval actually contains 'P' or more
- Simulated "Confidence" is % of successes out of 10000 trials

Partial table of simulation results using NIST k_2 (2-sided tolerance intervals), ready to graph

Two sided confidence

--------------------X-axis values----------------------

Y-axis

Proportion	N	Theoretical Confidence	%successes, actually$\geq$P	diff
0.90	5	75	75.29	.29
0.90	10	75	75.21	.21
0.90	30	75	74.80	-.20
0.90	100	75	75.15	.15
0.95	5	75	75.75	.75
0.95	10	75	75.16	.16
0.95	30	75	74.57	-.43
0.95	100	75	75.07	-.07

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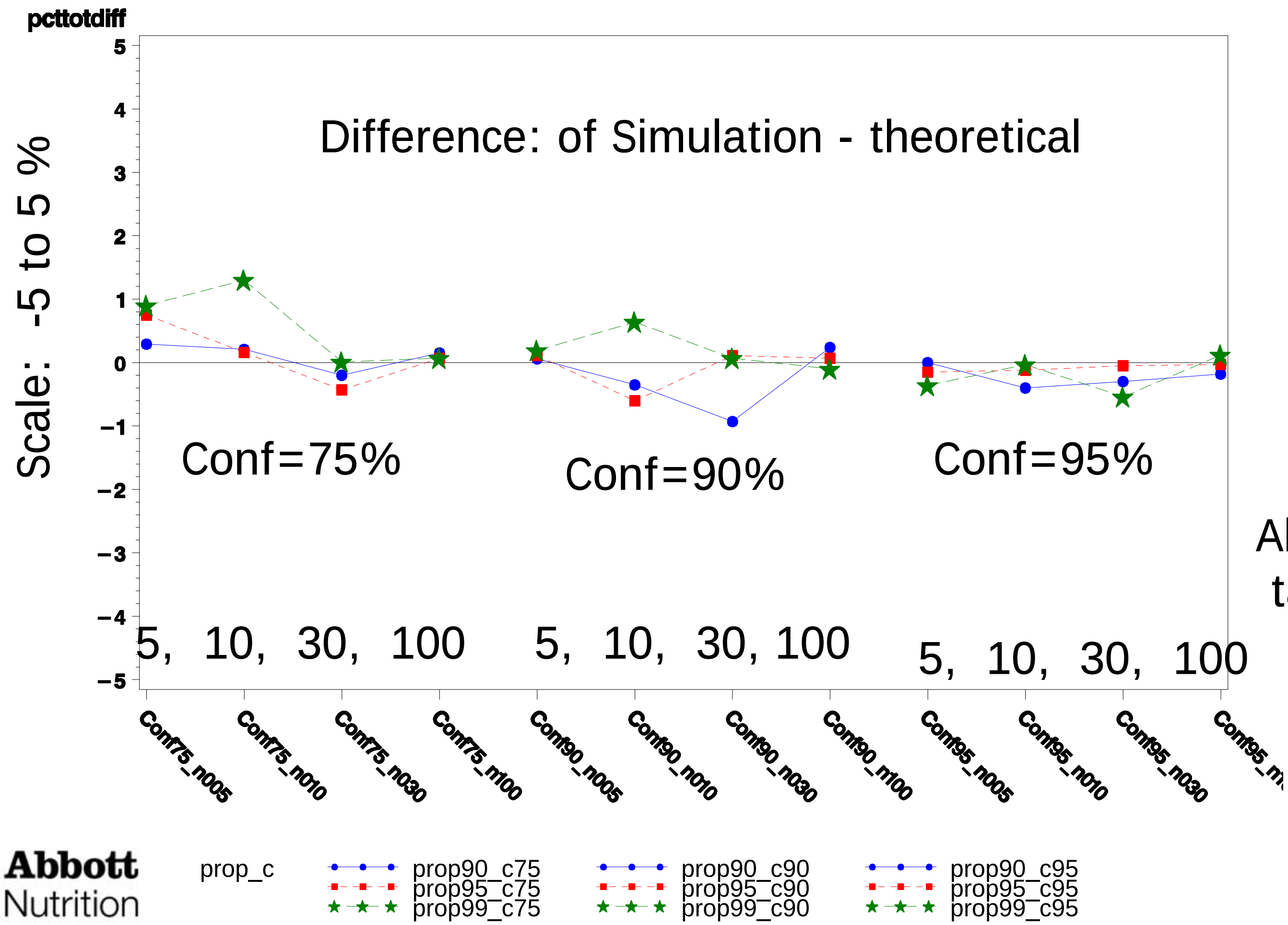
Graph of simulation results: NIST 2-sided tolerance confidences compared to simulation confidence



Blue: P=.90

Red: P=95

Green: P=.99

All within 1.5% of targeted confidence

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1-Sided Tolerance Interval

Who thought we'd need a special computation?!

Suppose the goal is to be λ% confident that at least P proportion of the data is ABOVE a lower limit. e.g. – you have a lower spec, no upper spec

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Three proposed 1-sided tolerance factors

- “I am l% confident that at least P of the data is above $m – k_1*s$”
  - 1. Adapt the 2-sided NIST formula for k_2, using z_{1-p} instead of $z_{(1-p)/2}$
  - 2. Use the 1-sided formula for k_1 on the NIST website
  - 3. Use a non-central t distribution, so

    K = k_noncent_t (also on NIST)

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Details on (2) k_1, NIST 1-sided calculation

k_1 NIST 1-sided

The calculation of an approximate *k* factor for one-sided tolerance intervals comes directly from the following set of formulas (Natrella, 1963):

$$k_1 = \frac{z_{1-p} + \sqrt{z_{1-p}^2 - ab}}{a}$$

$$a = 1 - \frac{z_{1-\gamma}^2}{2(N-1)}; \quad b = z_{1-p}^2 - \frac{z_{1-\gamma}^2}{N}$$

Details on Non-central t calculation (NIST)

The value of k can also be computed using the inverse cumulative distribution function for the non-central t distribution. This method may give more accurate results for small values of n.

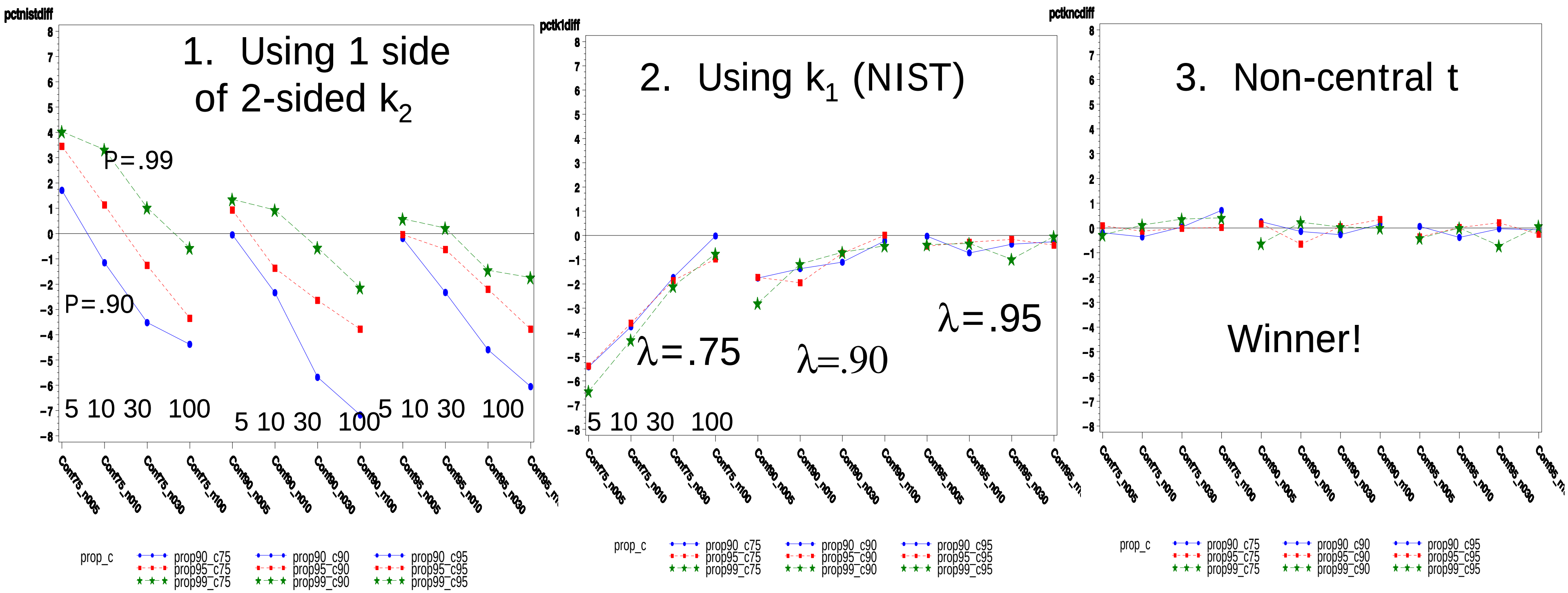
Let delta=z_{1-p}*sqrt(n), the noncentrality parameter

k_noncent_t=tinv(λ ,n-1,delta)/sqrt(n)

1-sided interval (lower bound):
Mean - k_noncent_t*s

Deviation from targeted confidence, Simulated minus theoretical

Same vertical scale, -8 to 8%



Blue P=.90; Red: P=.95; Green: P=.99

In conclusion

- The statement *“(m - K*s, m + K*s) contains at least a specified proportion of the data”* is only true a certain % of the time.
  - If K is a “tolerance” factor, we are assured that the interval contains at least proportion “P” of the data λ% of the time.
- One-sided tolerance intervals are not just an adaptation of the 2-sided formula, though this approach is better than nothing. NIST formulas are better, non-central t is best.

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The End

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Observations, #1

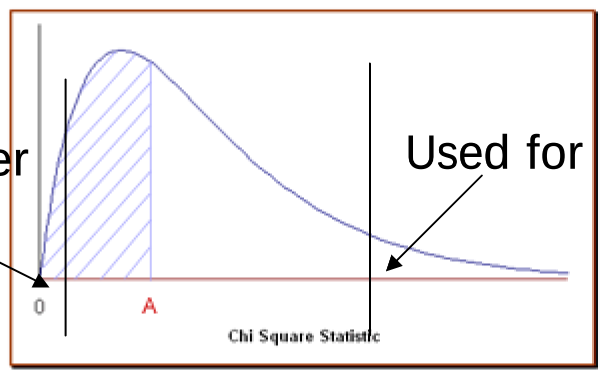
- Est. Variance of random variable X: s^2
- Est. Variance of mean of X: s^2/n
- Variance of random variable + mean = varsum = $s^2 + s^2/n$
  - Simplifies: varsum $=s^2(1+1/n)$
  - Sqrt for stdsum = s*sqrt(1+1/n)
- .95 'interval' for X +mean= Z*s*sqrt(1+1/n)
  - Put z under sqrt: **s*sqrt((1+1/n)*z²)**
- Adjust s so it is the upper λ% 1-sided confidence interval
  - Adjusted s = S*sqrt((n-1)/$\chi^2_{\lambda, n-1}$)
- Put adjusted s in expression for coverage:
  - S*sqrt((n-1)/$\chi^2_{\lambda, n-1}$) sqrt((1+1/n)*z^2))
- Simplify data under the radical sign
  - S*sqrt((n-1)(1+1/n)*z^2 /$\chi^2_{\lambda, n-1}$)
  - This is k_2

Observations, #2

- s^2 is an unbiased estimator of σ^2.
  - Unbiased refers to the mean of the s^2.
  - Distribution of s^2 is skewed, so mean is not median (50th %-ile).
  - Proportions/coverages are based on %-iles, not means.
- Distribution of $(n-1)*s^2/\sigma^2$ is χ^2_{n-1}
- A 2-sided confidence interval for σ is
  - $[s*(n-1)/\text{sqrt}(\chi^2_{.975,n-1}),\ s*(n-1)/\text{sqrt}(\chi^2_{.025,n-1})]$

RE: est k_2

Used for upper



Used for lower

Observations #3

- Regarding NIST k_1?? - From CRC ref., p. 293, the expression
  - Sqrt($2\mathcal{X}^2$) – sqrt(2n-1) is approx. normally distributed as the standard normal (mean=0, std=1)
  - Compute $\mathcal{X}^2_\alpha$ = .5*[sqrt(z_α+sqrt(2n-1)]^2

Recall - Confidence Interval of the mean

- Assume a normal distribution with a mean of μ and standard deviation of σ
  - μ estimated by m, sample mean
  - σ estimated by s, sample standard deviation (df=n-1)
- Confidence interval for μ
  - $m \pm t_{df,\text{“conf”}} * s/\text{sqrt}(n)$
- Calculate a 95% confidence interval, n=3
  - Mean = .36, std. = 1.55, t(2, .975) = 4.303
  - .36 +/- 4.303*1.55/sqrt(3) →
  - Confidence interval for μ: (-3.50, 4.22)

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Simulation, n=3; truth is: normal, $\mu=0$, $\sigma=1$

sample	x1	x2	x3	mean	std	m+/-s*t/rt(3) lower	upper	Contains “Truth”?
1	-0.80	-1.01	-1.71	-1.17	0.48	**-2.35**	**0.01**	yes
2	0.01	-0.98	2.06	0.36	1.55	**-3.50**	**4.22**	yes
3	-0.18	-0.98	-0.10	-0.42	0.49	**-1.63**	**0.80**	yes
4	0.37	1.51	-1.24	0.21	1.38	**-3.21**	**3.64**	yes
5	-0.06	0.68	-1.86	-0.41	1.31	**-3.66**	**2.84**	yes
6	1.58	-0.73	0.72	0.52	1.17	**-2.37**	**3.42**	yes
39	***0.33***	***0.49***	***0.30***	***0.37***	***0.10***	***0.12***	***0.62***	*No*
157	***-0.91***	***-0.68***	***-0.75***	***-0.78***	***0.12***	***-1.07***	***-0.49***	*No*
etc	etc							

~95% of the intervals will contain true μ

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