Weighted EWMA Charts for Monitoring Type I Censored Weibull Lifetimes

Shangjie Xu

Department of Statistics
University of California
Riverside, CA
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Outline

- Introduction of Lifetime Data and Weibull Distribution
- Monitoring for the Scale Parameter ($\eta$)
- Joint Monitoring for the Scale and Shape Parameters ($\eta$&$\beta$)
- Future Work
Lifetime Data

Motivation

- Lifetime of a product is a key criteria to evaluate its quality.
- Companies routinely perform life test on their products.

Two aspects

- Nonnormally distributed
- Right censored (time/expense limitations)
Weibull Distribution

Two-parameter Weibull probability density function (pdf) is

\[ f(t|\beta, \eta) = \frac{\beta}{\eta} \left( \frac{t}{\eta} \right)^{\beta-1} \exp\left( -\left( \frac{t}{\eta} \right)^\beta \right), \quad t > 0 \]

with cumulative distribution function (cdf)

\[ F(t|\beta, \eta) = 1 - \exp\left( -\left( \frac{t}{\eta} \right)^\beta \right) \]

where \( \beta > 0 \) is the shape parameter and \( \eta > 0 \) is the scale parameter.
Let $T \sim \text{Weibull}(\eta, \beta)$

- The censoring rate is $p_c = P(T \geq C) = \exp(-\left(\frac{C}{\eta}\right)^\beta)$

- The mean and variance are
  
  \[ E[T] = \eta \Gamma(1 + \frac{1}{\beta}), \quad Var[T] = \eta^2 \Gamma(1 + \frac{2}{\beta}) - (\eta \Gamma(1 + \frac{1}{\beta}))^2 \]

- For a fixed $\beta$, a decrease in the scale parameter $\eta$ (characteristic life) indicates a decrease in the mean lifetime of a product.

- Let $X = (T/\eta)^\beta$, then $X \sim \text{exp}(1)$
The Weibull lifetimes $T_{ij}$ are transformed or replaced by the conditional expected value (CEV).

$$X = \left( \frac{T}{\eta_0} \right)^\beta \sim \exp(1)$$

$$CE(X) = E(X|X > \left( \frac{C}{\eta_0} \right)^\beta) = \left( \frac{C}{\eta_0} \right)^\beta + 1 = -\log(p_c) + 1$$

Define $W_{ij} = \begin{cases} 
\left( \frac{T_{ij}}{\eta_0} \right)^\beta & \text{if } T_{ij} \leq C \\
CE(X) & \text{if } T_{ij} > C
\end{cases}$

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EWMA CEV Chart (Zhang and Chen 2004)

- The lower-sided EWMA CEV chart is
  \[ Q_i = \min((1 - \lambda)Q_{i-1} + \lambda \bar{W}_i, \omega_0), \quad Q_0 = \omega_0 \]
  where \( \bar{W}_i = (W_{i1} + W_{i2} + \cdots + W_{in})/n \), \( \lambda \in (0, 1] \) is the smoothing parameter, and \( \omega_0 = 1 \) is a reflecting barrier to increase the sensitivity of the chart.

- Alarm if \( Q_i < h \), where \( h \) is the control limit.
CUSUM Chart (Dickinson et al. 2014)\(^2\)

- Likelihood function for the \(ith\) batch sample \(T_{i1}, \ldots, T_{in}\) is

\[
L(\eta | T_i) = \prod_{j=1}^{n} f(T_{ij} | \eta)^{\delta_{ij}} [1 - F(T_{ij} | \eta)]^{1-\delta_{ij}}
\]

where \(\delta_{ij} = I(T_{ij} \text{ is an exact failure time})\)

- Lower-sided CUSUM for \(H_0: \eta = \eta_0\) v.s. \(H_1: \eta = \eta_1\) is:

\[
D_i = \max(0, D_{i-1} + z_i), \quad D_0 = 0
\]

where \(z_i = \log\left(\frac{L(\eta_1 | T_i)}{L(\eta_0 | T_i)}\right)\), and \(\eta_1 = (1 - d)\eta_0, \quad 0 < d < 1\)

CUSUM Chart (Dickinson et al. 2014)

- Lower-sided CUSUM (scaled)

\[ C_i = \min(0, C_{i-1} + \sum_{j=1}^{n} \left( \frac{T_{ij}}{\eta_0} \right)^\beta - k_i), \quad C_0 = 0 \]

where \( k_i = -\frac{r_i \beta \log \left( \frac{1}{1-d} \right)}{1 - (\frac{1}{1-d})^\beta} \); \( r_i \) = # of uncensored data in the \( ith \) batch sample

- Alarm if \( C_i < h \)
Assume $X_t$’s are independent Poisson observations with mean $\mu_t = n_t \theta$, where $n_t$ and $\theta$ are the size of the population at time $t$ and the incidence rate of a rare event.

- In health surveillance, detecting an increase in the incidence rate is often of interest
- $H_0 : \theta = \theta_0$ v.s. $H_1 : \theta > \theta_0$

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WEWMA Chart (Zhou et al. 2012)

- One-sided WEWMA chart tracking statistics

\[ R_{t,\lambda} = 2[Y_t(\hat{\theta}_t; \lambda) - Y_t(\theta_0; \lambda)] \]

\[ = 2 \sum_{i=0}^{t} \omega_{i,\lambda}[l_i(\hat{\theta}_t) - l_i(\theta_0)] \]

\[ \omega_{i,\lambda} = \lambda(1-\lambda)^{t-i}, \quad l_i(\theta) = X_i \log \theta - n_i \theta, \quad \tilde{\theta}_t = \arg\max_{\theta} Y_t(\theta; \lambda), \quad \hat{\theta}_t = \max(\tilde{\theta}_t, \theta_0) \]

- At time 0, define \((X_0, n_0) = (n_1 \theta_0, n_1)\) as a "pseudo sample"
- Alarm if \(R_{t,\lambda} > L\), \(L\) is a control limit
Proposed WEWMA Control Chart

Motivation

- The shapes of Poisson and Weibull distribution are similar.
- WEWMA outperforms EWMA and CUSUM for Poisson data.

Plan

- Apply the WEWMA chart scheme on censored Weibull data
- Consider the both censored and uncensored cases

⇒ Anticipate our proposed WEWMA chart can outperform CUSUM and EWMA CEV charts in the $ARL_1$ performance.
Weighted Log-likelihood (Censored)

At any time point $t$, consider the following exponentially weighted log-likelihood over batch samples of size $n$ from time 1 to $t$

$$Y_t(\eta; \beta, \lambda) = \sum_{i=1}^{t} \omega_i \lambda l_i(\eta)$$

where the log-likelihood for the batch sample at time $i$ is

$$l_i(\eta) = \sum_{j=1}^{n} \delta_{ij} \log(f(T_{ij}|\eta)) + \sum_{j=1}^{n} (1 - \delta_{ij}) \log(1 - F(T_{ij}|\eta)).$$
Then given the value of $\beta$ and $\lambda$, the (unconstrained) maximum weighted likelihood estimate (MWLE) of $\eta$ at time $t$ satisfies

$$\tilde{\eta}_t = \arg\max_{\eta} Y_t(\eta; \beta, \lambda)$$

By some simple algebra, we get

$$\tilde{\eta}^\beta_t = \begin{cases} \sum_{i=1}^{t} \omega_{i,\lambda} \sum_{j=1}^{n} T_{ij}^\beta & \text{if } \sum_{i=1}^{t} r_i \neq 0 \\ \sum_{i=1}^{t} \omega_{i,\lambda} r_i & \text{if } \sum_{i=1}^{t} r_i = 0 \end{cases}$$

where $r_i =$ # of uncensored data among the $ith$ batch sample. For our lower-sided chart, $\hat{\eta}_t = \min(\tilde{\eta}_t, \eta_0)$. 
Proposed WEWMA Control Chart

Then under hypothesis:

\[ H_0 : \eta = \eta_0 \text{ v.s. } H_1 : \eta < \eta_0, \]

The tracking statistics of our proposed chart is

\[
R_{t,\lambda} = 2[Y_t(\hat{\eta}_t; \beta, \lambda) - Y_t(\eta_0; \beta, \lambda)] = 2 \sum_{i=1}^{t} \omega_{i,\lambda}[l_i(\hat{\eta}_t) - l_i(\eta_0)]
\]

Alarm if \( R_{t,\lambda} > L \), \( L \) is the control limit.
Quick Review

- Interested hypothesis: \( H_0 : \eta = \eta_0 \) v.s. \( H_1 : \eta < \eta_0 \) (\( \beta \) is fixed)

- Existing works

  - EWMA CEV chart: \( Q_i = min((1 - \lambda)Q_{i-1} + \lambda W_i, \omega_0) \). Alarm if \( Q_i < L \).

  - CUSUM chart: \( C_i = min(0, C_{i-1} + \sum_{j=1}^{n} (\frac{T_{ij}}{\eta_0})^{\beta} + \frac{r_i \beta \log(\frac{1}{1-d})}{1-(\frac{1}{1-d})^{\beta}}) \). Alarm if \( C_i < L \).

- Proposed WEWMA chart: \( R_{t,\lambda} = 2 \sum_{i=1}^{t} \omega_i,\lambda [l_i(\hat{\eta}_t) - l_i(\eta_0)] \). Alarm if \( R_{t,\lambda} > L \).
Simulation Settings

- $\eta_0 = 1$
- sample size $n = 5$
- shape parameter $\beta = 0.5$ (decrease failure rate), 1 (constant failure rate), 3 (increase failure rate; normal), 5 (increase failure rate; skewed left)
- censoring rate $p_c = 0, 0.05, 0.50, 0.95$
- CUSUM design shift size $d = 0.15, 0.20, 0.30, 0.35$
- smoothing parameter $\lambda^4 = 0.05, 0.10, 0.20$
- \[ C = \eta_0(-\log(p_c))^{1/\beta} \]

\footnote{Followed by rule-of-thumb, $\lambda \in [0.05, 0.2]$}
Simulation Settings

For all the three charts, we have the same IC and OC simulation settings as follows:

- Control limits are obtained by bisection algorithm such that $\text{ARL}_0 \approx 370$ ($N_0 = 15,000$ in-control run lengths are used).

- $\text{ARL}_1$s are **steady-state**$^5$ ARLs with warm-up period equals to 50 ($N_1 = 50,000$ out-of-control run lengths are used).

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$^5$Steady-state ARL values are based on a delayed shift in the parameter; the chart has been IC for a warm-up period before a shift occurs.
Performance Comparison ($\beta = 5, p_c = 0, 0.15$)
Performance Comparison ($\beta = 1, p_c = 0, 0.15$)

\begin{align*}
\beta = 1, p_c &= 0 \\
\beta = 1, p_c &= 0.15
\end{align*}
Performance Comparison ($\beta = 1, p_c = 0.5, 0.95$)

- CUSUM (d = 0.30)
- EWMA CEV ($\lambda = 0.05$)
- EWMA CEV ($\lambda = 0.1$)
- WEWMA ($\lambda = 0.05$)
- WEWMA ($\lambda = 0.1$)
Findings

- For a great majority of cases, the proposed WEWMA method has smaller OC ARL values than both of the EWMA CEV and the CUSUM charts.

- Superiority still holds under the uncensored case.

- Exceptions that the CUSUM performs the best are the cases when $\beta \leq 1$ (least common case) and the censoring rate is high (can be avoided by $\uparrow C$).
Sometimes we may be interested in monitoring for a decrease in the mean lifetime when we only have a target value \((\eta_0, \beta_0)\).

Mainly two approaches: two-step approach and joint monitoring approach (McCracken and Chakraborti (2013)).

A two-step approach can have an inflated false alarm rate if adjustments are not made.

Joint monitoring approach is recommended.
The tracking statistic for a joint WEWMA chart for monitoring a decrease in the mean lifetime when we only have an IC target value for \((\eta, \beta)\) is:

\[
R_{t,\lambda} = 2[Y_t(\hat{\eta}_t, \hat{\beta}_t; \lambda) - Y_t(\eta_0, \beta_0; \lambda)] \\
= 2\sum_{i=1}^{t} \omega_{i,\lambda}[l_i(\hat{\eta}_t, \hat{\beta}_t) - l_i(\eta_0, \beta_0)].
\]

The MWLE of \((\eta, \beta)\) at time \(t\), say, \((\hat{\eta}_t, \hat{\beta}_t)\) is constrained in the set \(\{(\eta, \beta): \eta\Gamma(1 + 1/\beta) \leq \eta_0\Gamma(1 + 1/\beta_0)\}\).

The control chart alarms if \(R_{t,\lambda} > L\), \(L\) is the control limit.
Joint CUSUM Control Chart

Let \((\eta_1, \beta_1)\) be the specified OC value of \((\eta, \beta)\), then the joint CUSUM chart is based on

\[
Z_i = \log \left( \frac{L(\eta_1, \beta_1|T_i)}{L(\eta_0, \beta_0|T_i)} \right)
\]

\[
= \sum_{j=1}^{n} \delta_{ij} \log \left( \frac{\frac{\beta_1}{\eta_1} \left( \frac{T_{ij}}{\eta_1} \right)^{\beta_1-1}}{\frac{\beta_0}{\eta_0} \left( \frac{T_{ij}}{\eta_0} \right)^{\beta_0-1}} \right) + \sum_{j=1}^{n} \left( \left( \frac{T_{ij}}{\eta_0} \right)^{\beta_0} - \left( \frac{T_{ij}}{\eta_1} \right)^{\beta_1} \right).
\]

where \(\eta_1 \Gamma(1 + 1/\beta_1) = (1 - D_1)\eta_0 \Gamma(1 + 1/\beta_0)\), \(D_1\) is the design relative shift size in the mean lifetime.
Simulation Settings

- $\eta_0 = 1$, $\beta_0 = 5$
- sample size $n = 5$
- censoring rate $p_c \in \{0, 0.50\}$
- joint CUSUM alternatives $D_1 \in \{0.1, 0.2\}$, $\beta_1 \in \{4, 6\}$
- smoothing parameter $\lambda \in \{0.10, 0.20\}$
- OC simulation settings:
  - $D \in \{0.025, 0.05, 0.10, 0.15, 0.20, 0.25\}$, and
  - $\beta \in \{4, 5, 6\}$. 
## Performance Comparison

<table>
<thead>
<tr>
<th>$D$</th>
<th>$\beta$</th>
<th>$p_c = 0$</th>
<th>$p_c = 0.5$</th>
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<tbody>
<tr>
<td></td>
<td>Joint WEWMA</td>
<td>Joint CUSUM</td>
<td>Joint WEWMA</td>
</tr>
<tr>
<td>0</td>
<td>$\lambda = 0.1$</td>
<td>$\lambda = 0.2$</td>
<td>$\lambda = 0.1$</td>
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<tr>
<td>5</td>
<td>370.10</td>
<td>374.82</td>
<td>371.26</td>
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<td>0.025</td>
<td>6</td>
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<td></td>
<td>5</td>
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<td>126.69</td>
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<td></td>
<td>4</td>
<td><strong>16.90</strong></td>
<td><strong>18.42</strong></td>
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<td>6</td>
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<td>6</td>
<td>7.23</td>
<td><strong>6.32</strong></td>
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<td>9.02</td>
<td><strong>8.72</strong></td>
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<td>4.88</td>
<td><strong>3.95</strong></td>
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<td><strong>4.45</strong></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>5.42</td>
<td><strong>4.80</strong></td>
</tr>
</tbody>
</table>

*In each row, the lowest OC ARLs are set in bold and italic. The second lowest OC ARLs are set in bold only.*
Findings

- The joint CUSUM chart performs better when the actual OC scenario is close to its design alternatives.
- The joint WEWMA performs better when the disparity between the design alternatives of the joint CUSUM and the actual OC scenarios increases.
- When the joint CUSUM performs better, the joint WEWMA is usually very competitive.
Summary

- Interest: Detecting a decrease in right-censored Weibull lifetimes.
- $\beta$ is fixed: WEWMA > CUSUM > EWMA CEV
- $\beta$ is not fixed: joint WEWMA > joint CUSUM

Future Work

- Phase I analysis: when IC parameters are unknown, we can determine the phase I study sample size to control the accuracy of the conditional IC ARL (Jeske (2016)).
- Extend to time-varying batch sizes: use bootstrap method to control the conditional performance of control charts (Gandy and Kvaloy (2013)).
Thank you!