Weighted EWMA Charts for Monitoring Type I Censored Weibull Lifetimes

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Outline

- Introduction of Lifetime Data and Weibull Distribution
- Monitoring for the Scale Parameter (η)
- Joint Monitoring for the Scale and Shape Parameters (η&β)
- Future Work

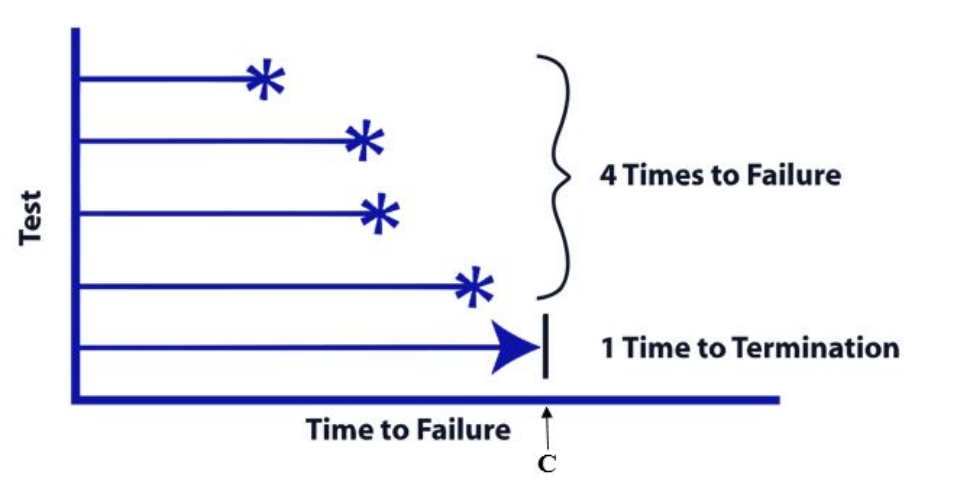
Lifetime Data

Motivation

- Lifetime of a product is a key criteria to evaluate its quality.
- Companies routinely perform life test on their products.

Two aspects

- Nonnormally distributed
- Right censored (time/expense limitations)



Weibull Distribution

Two-parameter Weibull probability density function (pdf) is

$$f(t|\beta,\eta) = \frac{\beta}{\eta}(\frac{t}{\eta})^{\beta-1} exp(-(\frac{t}{\eta})^{\beta}), t > 0$$

with cumulative distribution function (cdf)

$$F(t|\beta,\eta) = 1 - exp(-(\frac{t}{\eta})^{\beta})$$

where $\beta > 0$ is the shape parameter and $\eta > 0$ is the scale parameter.

Weibull Distribution

Let $T \sim \text{Weibull}(\eta, \beta)$

- The **censoring rate** is $p_c = P(T \geq C) = exp(-(\frac{C}{\eta})^{\beta})$
- The mean and variance are
  $E[T] = \eta\Gamma(1 + \frac{1}{\beta})$, $Var[T] = \eta^2\Gamma(1 + \frac{2}{\beta}) - (\eta\Gamma(1 + \frac{1}{\beta}))^2$
- For a fixed β, a decrease in the scale parameter η (characteristic life) indicates a decrease in the mean lifetime of a product.
- Let $X = (T/\eta)^{\beta}$, then $X \sim exp(1)$

EWMA CEV Chart (Zhang and Chen 2004)[1]

- The Weibull lifetimes T_{ij} are transformed or replaced by the conditional expected value (CEV).

$$X = (T/\eta_0)^\beta \sim exp(1)$$

$$CE(X) = E(X|X > (C/\eta_0)^\beta) = (\frac{C}{\eta_0})^\beta + 1 = -log(p_c) + 1$$

$$\text{Define } W_{ij} = \begin{cases} (\frac{T_{ij}}{\eta_0})^\beta & \text{if } T_{ij} \leq C \\ CE(X) & \text{if } T_{ij} > C \end{cases}$$

[1] Zhang, L. and Chen G. *EWMA charts for monitoring the mean of censored Weibull lifetimes*. Journal of Quality Technology, Vol. 36, No. 3, 2004.

EWMA CEV Chart (Zhang and Chen 2004)

- The lower-sided EWMA CEV chart is

$$Q_i = min((1-\lambda)Q_{i-1} + \lambda\overline{W_i}, \omega_0), Q_0 = \omega_0$$

where $\overline{W_i} = (W_{i1} + W_{i2} + \cdots + W_{in})/n, \lambda \in (0, 1]$ is the smoothing parameter, and $\omega_0 = 1$ is a reflecting barrier to increase the sensitivity of the chart.

- Alarm if $Q_i < h$, where h is the control limit.

CUSUM Chart (Dickinson et al.2014)[2]

- Likelihood function for the *ith* batch sample $T_{i1}, ..., T_{in}$ is

$$L(\eta|\boldsymbol{T_i}) = \prod_{j=1}^{n} f(T_{ij}|\eta)^{\delta_{ij}}[1 - F(T_{ij}|\eta)]^{1-\delta_{ij}}$$

where δ_{ij}=I(T_{ij} is an exact failure time)

- Lower-sided CUSUM for $H_0 : \eta = \eta_0$ v.s. $H_1 : \eta = \eta_1$ is:

$$D_i = max(0, D_{i-1} + z_i), D_0 = 0$$

where $z_i = log(\frac{L(\eta_1|\boldsymbol{T_i})}{L(\eta_0|\boldsymbol{T_i})})$, and $\eta_1 = (1-d)\eta_0$, $0 < d < 1$

[2] Rebecca M. Dickinson, Denisa A. Olteanu Roberts, Anne R. Driscoll, William Woodall, and G. Geoffrey Vining. *CUSUM charts for monitoring the characteristic life of censored Weibull lifetimes*. Journal of Quality Technology, Vol. 46, No. 4, 2014.

CUSUM Chart (Dickinson et al.2014)

- Lower-sided CUSUM (scaled)

$$C_i = min(0, C_{i-1} + \sum_{j=1}^{n}(\frac{T_{ij}}{\eta_0})^{\beta} - k_i), C_0 = 0$$

where $k_i = -\frac{r_i \beta log(\frac{1}{1-d})}{1-(\frac{1}{1-d})^{\beta}}$; r_i= # of uncensored data in the *ith* batch sample

- Alarm if $C_i < h$

WEWMA Chart (Zhou et al.2012)[3]

Assume X_t's are independent Poisson observations with mean $\mu_t = n_t\theta$, where n_t and θ are the size of the population at time t and the incidence rate of a rare event.

- In health surveillance, detecting an increase in the incidence rate is often of interest
- $H_0 : \theta = \theta_0$ v.s. $H_1 : \theta > \theta_0$

[3] Qin Zhou, Changliang Zou, Zhaojun Wang, and Wei Jiang. *Likelihood-based EWMA charts for monitoring Poisson count data with time-varying sample sizes*. Journal of the American Statistical Association, Vol. 107, No. 499, 2012.

WEWMA Chart (Zhou et al.2012)

- One-sided WEWMA chart tracking statistics

$$R_{t,\lambda} = 2[Y_t(\hat{\theta}_t;\lambda) - Y_t(\theta_0;\lambda)]$$
$$= 2\sum_{i=0}^{t} \omega_{i,\lambda}[l_i(\hat{\theta}_t) - l_i(\theta_0)]$$

$$\omega_{i,\lambda} = \lambda(1-\lambda)^{t-i},\ l_i(\theta) = X_i log\theta - n_i\theta,\ \tilde{\theta}_t = \underset{\theta}{\text{argmax}} Y_t(\theta;\lambda),\ \hat{\theta}_t = max(\tilde{\theta}_t, \theta_0)$$

- At time 0, define $(X_0, n_0) = (n_1\theta_0, n_1)$ as a "pseudo sample"
- Alarm if $R_{t,\lambda} > L$, L is a control limit

Proposed WEWMA Control Chart

Motivation

- The shapes of Poisson and Weibull distribution are similar.
- WEWMA outperforms EWMA and CUSUM for Poisson data.

Plan

- Apply the WEWMA chart scheme on censored Weibull data
- Consider the both censored and uncensored cases

$\Rightarrow$ Anticipate our proposed WEWMA chart can outperform CUSUM and EWMA CEV charts in the ARL_1 performance.

Weighted Log-likelihood (Censored)

At any time point t, consider the following exponentially weighted log-likelihood over batch samples of size n from time 1 to t

$$Y_t(\eta; \beta, \lambda) = \sum_{i=1}^{t} \omega_{i,\lambda} l_i(\eta)$$

where the log-likelihood for the batch sample at time i is

$$l_i(\eta) = \sum_{j=1}^{n} \delta_{ij} log(f(T_{ij}|\eta)) + \sum_{j=1}^{n} (1 - \delta_{ij}) log(1 - F(T_{ij}|\eta)).$$

Weighted MLE (Censored)

Then given the value of β and λ, the (unconstrained) maximum weighted likelihood estimate (MWLE) of η at time t satisfies

$$\tilde{\eta}_t = \underset{\eta}{\operatorname{argmax}} Y_t(\eta; \beta, \lambda)$$

By some simple algebra, we get

$$\tilde{\eta}_t^{\beta} = \begin{cases} \dfrac{\sum_{i=1}^{t} \omega_{i,\lambda} \sum_{j=1}^{n} T_{ij}^{\beta}}{\sum_{i=1}^{t} \omega_{i,\lambda} r_i} & \text{if } \sum_{i=1}^{t} r_i \neq 0 \\ \eta_0^{\beta} & \text{if } \sum_{i=1}^{t} r_i = 0 \end{cases}$$

where $r_i = \#$ of uncensored data among the *ith* batch sample.
For our lower-sided chart, $\hat{\eta}_t = min(\tilde{\eta}_t, \eta_0)$.

Proposed WEWMA Control Chart

Then under hypothesis:

$$H_0 : \eta = \eta_0 \text{ v.s. } H_1 : \eta < \eta_0,$$

The tracking statistics of our proposed chart is

$$\begin{aligned} R_{t,\lambda} &= 2[Y_t(\hat{\eta}_t; \beta, \lambda) - Y_t(\eta_0; \beta, \lambda)] \\ &= 2\sum_{i=1}^{t} \omega_{i,\lambda}[l_i(\hat{\eta}_t) - l_i(\eta_0)] \end{aligned}$$

Alarm if $R_{t,\lambda} > L$, L is the control limit.

Quick Review

- Interested hypothesis: $H_0 : \eta = \eta_0$ v.s. $H_1 : \eta < \eta_0$ (β is fixed)
- Existing works
  - EWMA CEV chart: $Q_i = min((1-\lambda)Q_{i-1} + \lambda\overline{W_i}, \omega_0)$. Alarm if $Q_i < L$.
  - CUSUM chart: $C_i = min(0, C_{i-1} + \sum_{j=1}^{n}(\frac{T_{ij}}{\eta_0})^\beta + \frac{r_i\beta log(\frac{1}{1-d})}{1-(\frac{1}{1-d})^\beta})$.

    Alarm if $C_i < L$.
- Proposed WEWMA chart: $R_{t,\lambda} = 2\sum_{i=1}^{t}\omega_{i,\lambda}[l_i(\hat{\eta}_t) - l_i(\eta_0)]$.

  Alarm if $R_{t,\lambda} > L$.

Simulation Settings

- $\eta_0 = 1$
- sample size $n = 5$
- shape parameter $\beta = 0.5$ (decrease failure rate), 1 (constant failure rate), 3 (increase failure rate; normal), 5 (increase failure rate; skewed left)
- censoring rate $p_c = 0, 0.05, 0.50, 0.95$
- CUSUM design shift size $d = 0.15, 0.20, 0.30, 0.35$
- smoothing parameter λ[4] $= 0.05, 0.10, 0.20$
- $C = \eta_0(-log(p_c))^{1/\beta}$

[4] Followed by rule-of-thumb, $\lambda \in [0.05, 0.2]$

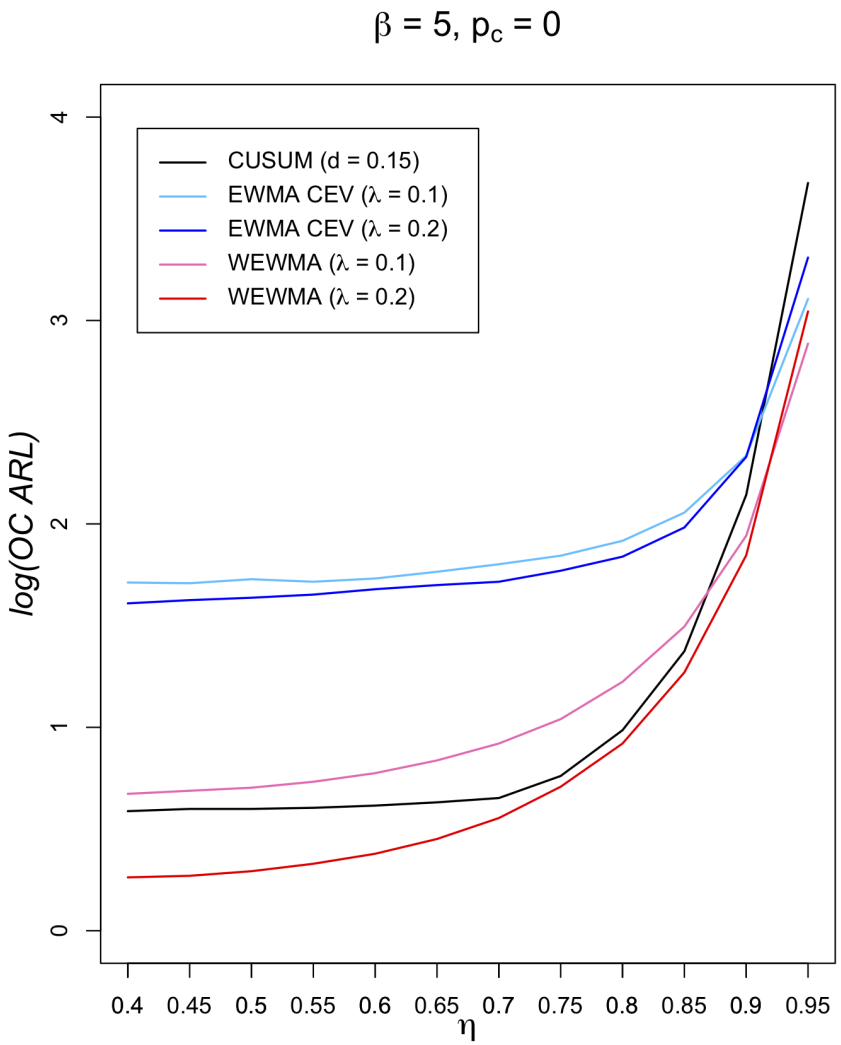
Simulation Settings

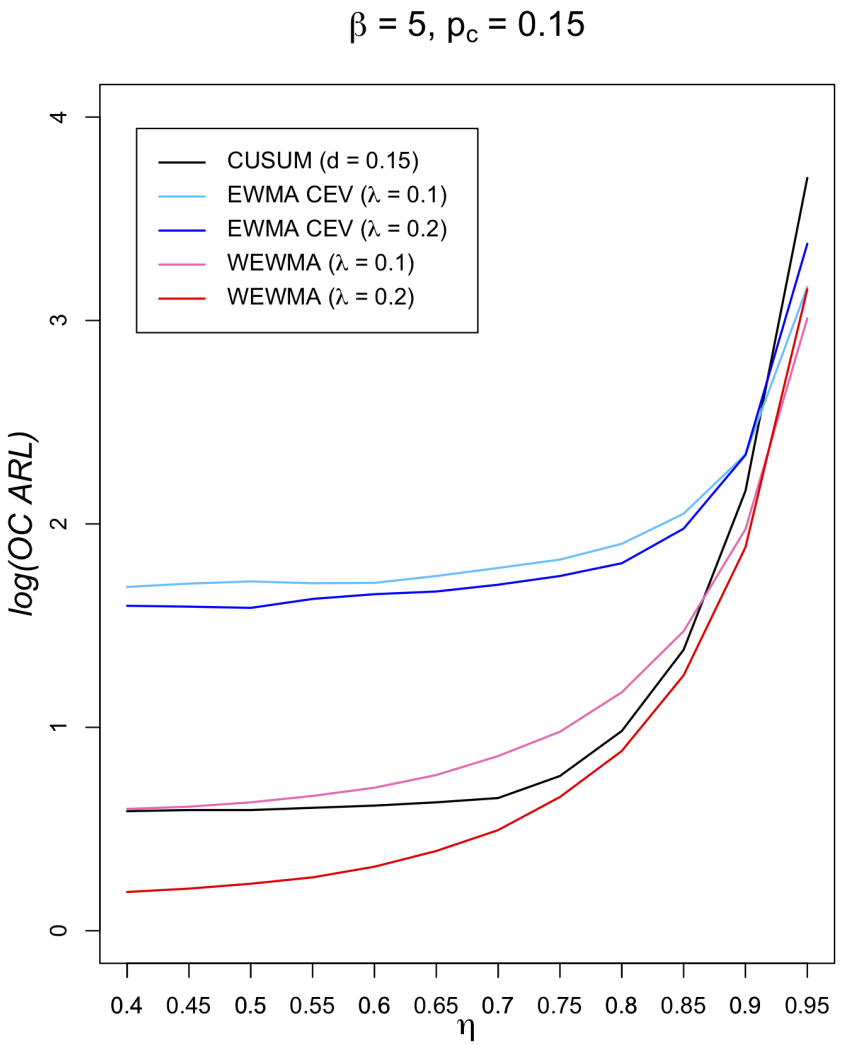
For all the three charts, we have the same IC and OC simulation settings as follows:

- Control limits are obtained by bisection algorithm such that $\text{ARL}_0 \approx 370$ ($N_0 = 15,000$ in-control run lengths are used).
- ARL_1s are **steady-state**[5] ARLs with warm-up period equals to 50 ($N_1 = 50,000$ out-of-control run lengths are used).

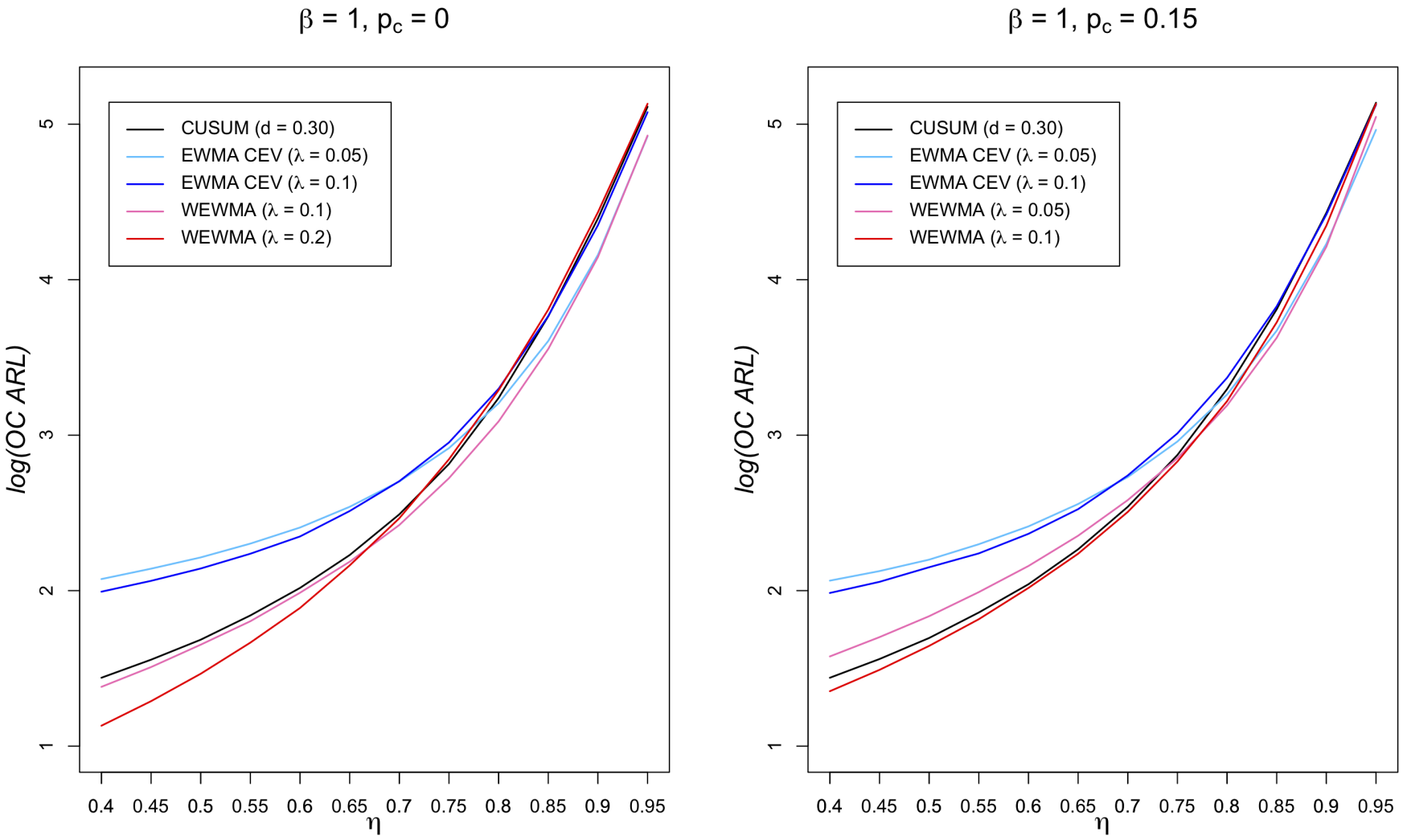
[5] Steady-state ARL values are based on a delayed shift in the parameter; the chart has been IC for a warm-up period before a shift occurs.

Performance Comparison ($\beta = 5, p_c = 0, 0.15$)

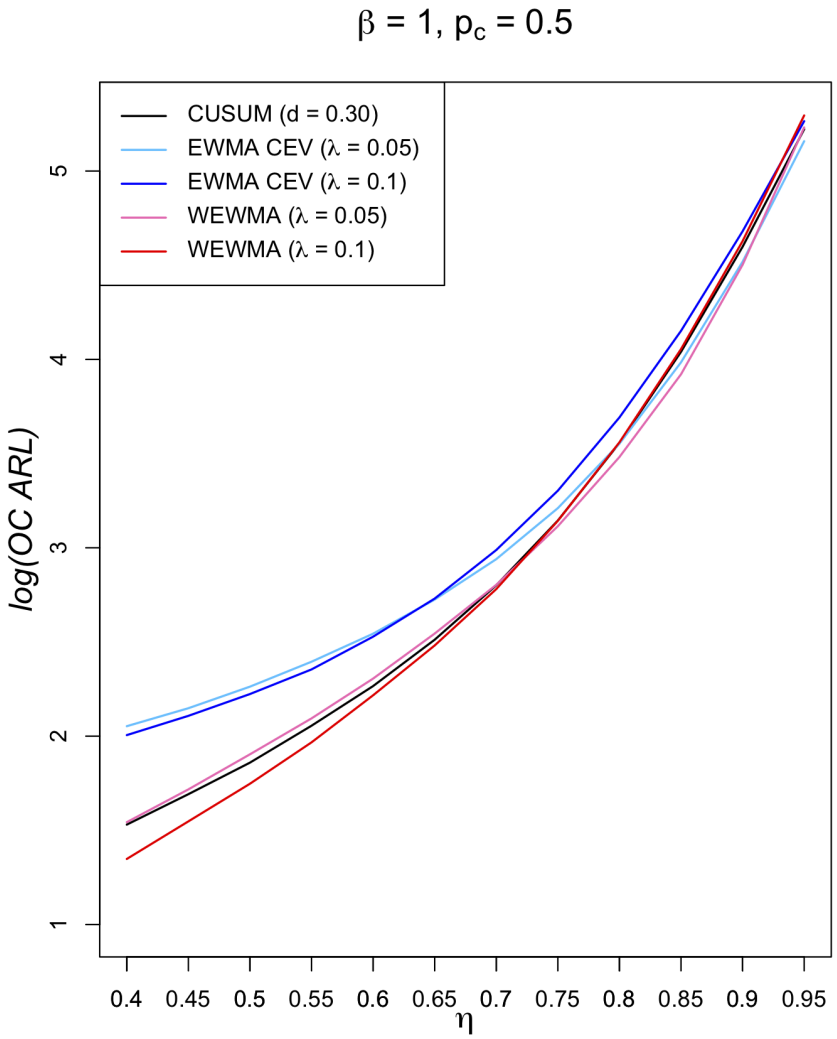


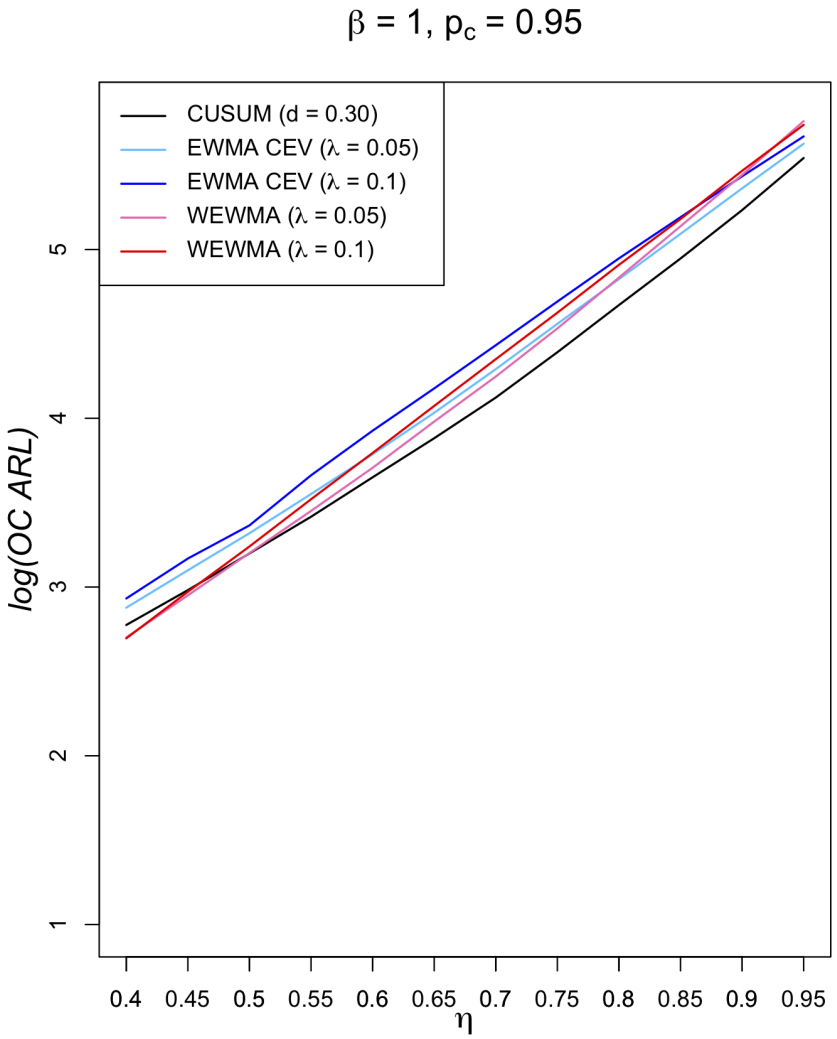


Performance Comparison ($\beta = 1, p_c = 0, 0.15$)



Performance Comparison ($\beta = 1, p_c = 0.5, 0.95$)





Findings

- For a great majority of cases, the proposed WEWMA method has smaller OC ARL values than both of the EWMA CEV and the CUSUM charts.
- Superiority still holds under the uncensored case.
- Exceptions that the CUSUM performs the best are the cases when $\beta \leq 1$ (least common case) and the censoring rate is high (can be avoided by $\uparrow C$).

Joint Monitoring for η and β

- Sometimes we may be interested in monitoring for a decrease in the mean lifetime when we only have a target value (η_0, β_0).
- Mainly two approaches: two-step approach and joint monitoring approach (McCracken and Chakraborti (2013)).
- A two-step approach can have an inflated false alarm rate if adjustments are not made.
- Joint monitoring approach is recommended.

Proposed Joint WEWMA Control Chart

The tracking statistic for a joint WEWMA chart for monitoring a decrease in the mean lifetime when we only have an IC target value for (η, β) is:

$$\begin{aligned} R_{t,\lambda} &= 2[Y_t(\hat{\eta}_t, \hat{\beta}_t; \lambda) - Y_t(\eta_0, \beta_0; \lambda)] \\ &= 2\sum_{i=1}^{t} \omega_{i,\lambda}[l_i(\hat{\eta}_t, \hat{\beta}_t) - l_i(\eta_0, \beta_0)]. \end{aligned}$$

The MWLE of (η, β) at time t, say, $(\hat{\eta}_t, \hat{\beta}_t)$ is constrained in the set $\{(\eta, \beta)\colon \eta\Gamma(1 + 1/\beta) \leq \eta_0\Gamma(1 + 1/\beta_0)\}$.

The control chart alarms if $R_{t,\lambda} > L$, L is the control limit.

Joint CUSUM Control Chart

Let (η_1, β_1) be the specified OC value of (η, β), then the joint CUSUM chart is based on

$$Z_i = log\left(\frac{L(\eta_1, \beta_1|\mathbf{T_i})}{L(\eta_0, \beta_0|\mathbf{T_i})}\right)$$
$$= \sum_{j=1}^{n} \delta_{ij} log\left(\frac{\frac{\beta_1}{\eta_1}\left(\frac{T_{ij}}{\eta_1}\right)^{\beta_1-1}}{\frac{\beta_0}{\eta_0}\left(\frac{T_{ij}}{\eta_0}\right)^{\beta_0-1}}\right) + \sum_{j=1}^{n}\left(\left(\frac{T_{ij}}{\eta_0}\right)^{\beta_0} - \left(\frac{T_{ij}}{\eta_1}\right)^{\beta_1}\right).$$

where $\eta_1\Gamma(1+1/\beta_1) = (1-D_1)\eta_0\Gamma(1+1/\beta_0)$, D_1 is the design relative shift size in the mean lifetime.

Simulation Settings

- $\eta_0 = 1$, $\beta_0 = 5$
- sample size $n = 5$
- censoring rate $p_c \in \{0, 0.50\}$
- joint CUSUM alternatives $D_1 \in \{0.1, 0.2\}$, $\beta_1 \in \{4, 6\}$
- smoothing parameter $\lambda \in \{0.10, 0.20\}$
- OC simulation settings:
  $D \in \{0.025,\ 0.05,\ 0.10,\ 0.15,\ 0.20,\ 0.25\}$, and
  $\beta \in \{4,\ 5,\ 6\}$.

Performance Comparison

		$p_c = 0$						$p_c = 0.5$					
		Joint WEWMA		Joint CUSUM				Joint WEWMA		Joint CUSUM			
D	β	$\lambda = 0.1$	$\lambda = 0.2$	$D_1 = 0.1$ $\beta_1 = 4$	$D_1 = 0.1$ $\beta_1 = 6$	$D_1 = 0.2$ $\beta_1 = 4$	$D_1 = 0.2$ $\beta_1 = 6$	$\lambda = 0.1$	$\lambda = 0.2$	$D_1 = 0.1$ $\beta_1 = 4$	$D_1 = 0.1$ $\beta_1 = 6$	$D_1 = 0.2$ $\beta_1 = 4$	$D_1 = 0.2$ $\beta_1 = 6$
0	5	370.10	374.82	371.26	370.85	372.16	372.04	374.58	371.21	373.10	373.96	373.94	370.49
0.025	6	***19.53***	**25.48**	475.20	34.56	344.25	171.49	***58.27***	**81.44**	723.09	81.85	516.49	222.34
	5	***91.77***	126.69	113.43	**104.94**	144.26	159.14	123.00	148.98	**121.74**	***112.76***	157.82	163.89
	4	***16.90***	**18.42**	28.60	394.43	64.10	151.45	**30.36**	36.26	***28.53***	190.47	48.84	120.94
0.05	6	***12.68***	13.44	88.37	**13.10**	95.16	60.19	***26.63***	34.42	143.10	**27.00**	157.74	80.24
	5	***27.07***	35.95	39.66	**35.82**	56.76	69.59	***39.93***	51.97	45.44	**40.76**	66.86	73.36
	4	***16.27***	17.68	**17.48**	150.05	34.16	80.51	**19.81**	23.91	***17.41***	78.86	27.98	65.69
0.1	6	7.23	**6.32**	11.82	***4.96***	11.92	10.37	8.93	**8.90**	15.30	***7.42***	18.86	13.72
	5	9.02	**8.72**	10.25	***8.60***	11.91	15.53	**11.07**	11.68	11.64	***10.81***	15.03	17.06
	4	9.31	**9.29**	***8.06***	28.77	11.24	24.40	**9.52**	9.72	***8.19***	20.33	10.80	20.87
0.15	6	4.88	**3.95**	5.34	***3.39***	4.32	3.59	4.79	***4.07***	5.90	**4.08**	5.29	4.35
	5	5.26	***4.45***	5.19	**4.56**	4.62	5.25	5.63	***5.02***	5.58	**5.44**	5.47	5.82
	4	5.42	***4.80***	**4.82**	9.65	5.03	8.85	5.53	**5.05**	***4.96***	8.99	5.21	8.20

*In each row, the lowest OC ARLs are set in bold and italic. The second lowest OC ARLs are set in bold only.

Findings

- The joint CUSUM chart performs better when the actual OC scenario is close to its design alternatives.
- The joint WEWMA performs better when the disparity between the design alternatives of the joint CUSUM and the actual OC scenarios increases.
- When the joint CUSUM performs better, the joint WEWMA is usually very competitive.

Summary and Future Work

Summary

- Interest: Detecting a decrease in right-censored Weibull lifetimes.
- β is fixed: WEWMA > CUSUM > EWMA CEV
- β is not fixed: joint WEWMA > joint CUSUM

Future Work

- Phase I analysis: when IC parameters are unknown, we can determine the phase I study sample size to control the accuracy of the conditional IC ARL (Jeske (2016)).
- Extend to time-varying batch sizes: use bootstrap method to control the conditional performance of control charts (Gandy and Kvaloy (2013)).

Thank you!